Applications of Fourier Transforms in Solving Differential Equations

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History

- Daniel Bernoulli
  - Known for the Bernoulli Principle
  - Solved the wave equation in 1720’s using techniques Fourier later would employ
  - Worked with Euler in St. Petersburg
History

- Jean le Rond d'Alembert
  - French mathematician
  - Worked with Euler and Bernoulli on the wave equation
  - “Found” the Cauchy-Riemann equations decades before Cauchy or Riemann did
History

- Joseph Fourier
  - French mathematician
  - Served in Egypt while in Napoleon’s Army
  - Solved the heat equation and won award for it in 1811, published work in 1822 in *Théorie analytique de la chaleur*
Erwin Schrödinger
- Vienna-born physicist
- Artilleryman in WWI
- Best known for “Schrödinger’s Cat” and the Schrödinger Equation
- Won the Nobel Prize in 1933
- Worked with Einstein
- Fled Germany and Austria several times after “insulting” the Nazis
The following concepts were used

Fourier Coefficients

\[ C_n = \frac{1}{P} \int_{0}^{P} f(x) e^{\frac{-2\pi inx}{P}} \, dx \]

Fourier Series

\[ FS(f)(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{2\pi inx}{P}} \]

Superposition Principle

The Superposition Principle states that for a linear system, the sum of solutions for that system is also a solution.
Concepts

- Orthogonality of Sines

Given

\[ f(x) = \sin \left( \frac{n \pi x}{L} \right); \ n \geq 1, \]

\[ \frac{2}{L} \int_{0}^{L} \sin \left( \frac{m \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right) dx = 0 \]

for \( m \neq n \)
Concepts

- Proof of Orthogonality

\[
\frac{2}{L} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \left[ \frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{2(m-n)\pi} - \frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{2(m+n)\pi} \right]_0^L
\]

\[
= \frac{2}{L} \left( \frac{\sin\left(\frac{(m-n)\pi}{L}\right)}{2(m-n)\pi} - \frac{\sin\left(\frac{(m+n)\pi}{L}\right)}{2(m+n)\pi} \right) - \frac{2}{L} (\sin(0) - \sin(0))
\]

\[
= \frac{2}{L} (0 - 0) - \frac{2}{L} (0 - 0)
\]

\[
= 0
\]
Wave Equation

- People studied the wave equation because they were intrigued by string instruments.
- Bernoulli defined nodes and frequencies of oscillation.
Derivation of Wave Equation

- To solve the wave equation, we assume that it is a separable equation:

\[ u(x,t) = X(x)T(t) \]

The Wave Equation

\[ \frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2} \]

Boundary conditions:
\[ u(0,t) = 0 \quad u(L,t) = 0 \]

Initial Conditions:
\[ u(x,0) = f(x) \quad \frac{\partial u}{\partial t}(x,0) = g(x) \]
Derivation

\[ \frac{\partial u}{\partial t} = XT' \]
\[ \frac{\partial^2 u}{\partial t^2} = XT'' \]

\[ \frac{\partial u}{\partial x} = X'T \]
\[ \frac{\partial^2 u}{\partial x^2} = X''T \]

\[ \Rightarrow \frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow T''X = \omega^2 X''T \]

By Separation of Variables we get:

\[ \frac{T''}{\omega^2 T} = \frac{X''}{X} = -\lambda \]

because the only way the two sides will be equal is if they equal a constant
Solve for $X$ as in the heat equation and you get

$$X = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right)$$

Solve for $T$

$$T''' = -\omega^2 \lambda T \quad \Rightarrow \quad T''' + \omega^2 \lambda T = 0$$

$$(D^2 + \omega^2 \lambda) T = 0 \quad \Rightarrow \quad D = \pm \left(\frac{\omega n \pi}{L}\right) i = \pm \gamma_n \quad n \in \mathbb{Z}$$

Therefore: $T = D_1 \cos(\gamma_n t) + D_2 \sin(\gamma_n t)$

Thus we get

$$u(x, t) = X(x) T(t)$$

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right) \left( D_n \cos(\gamma_n t) + E_n \sin(\gamma_n t) \right) \quad \text{by Superposition Principle}$$

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \left( D_n \cos(\gamma_n t) + E_n \sin(\gamma_n t) \right)$$
Using the Initial Conditions: \( u(x,0) = f(x) \)

\[
\begin{align*}
u(x,0) &= \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{L} x \right) \left( D_n \cos(\gamma_n 0) + E_n \sin(\gamma_n 0) \right) = \sum_{n=1}^{\infty} D_n \sin \left( \frac{n\pi}{L} x \right) = f(x) \\
\text{where } D_n &= \frac{2}{L} \int_0^L f(x) \sin(\sqrt{\lambda_n} x) \, dx \\
\frac{\partial u}{\partial t}(x,0) &= g(x) \\
\frac{\partial u}{\partial t}(x,t) &= \sum_{n=1}^{\infty} \gamma_n \sin \left( \frac{n\pi}{L} x \right) \left( -D_n \sin(\gamma_n t) + E_n \cos(\gamma_n t) \right) \\
\frac{\partial u}{\partial t}(x,0) &= \sum_{n=1}^{\infty} \gamma_n \sin \left( \frac{n\pi}{L} x \right) \left( -D_n \sin(\gamma_n 0) + E_n \cos(\gamma_n 0) \right) = \sum_{n=1}^{\infty} E_n \sin \left( \frac{n\pi}{L} x \right) = g(x) \\
\text{where } E_n &= \frac{2}{\gamma_n L} \int_0^L g(x) \sin(\sqrt{\lambda_n} x) \, dx
\end{align*}
\]

**Final Solution**

\[
\begin{align*}
u(x,t) &= \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{L} x \right) \left( D_n \cos(\gamma_n t) + E_n \sin(\gamma_n t) \right) \\
\text{where } D_n &= \frac{2}{L} \int_0^L f(x) \sin(\sqrt{\lambda_n} x) \, dx \quad \text{and} \quad E_n = \frac{2}{\gamma_n L} \int_0^L g(x) \sin(\sqrt{\lambda_n} x) \, dx
\end{align*}
\]
Unfiltered Wave Equation

![Graph of the Unfiltered Wave Equation](image-url)
Filtered Wave Equation
The reason we solve the heat equation is to find out what happens to a length of metal/plastic when heat is applied (the temperature distribution over time)
The assumption made by Bernoulli and adopted by Fourier was that the equation $u(x,t)$ was separable:

$$u(x,t) = X(x)T(t)$$

The Heat Equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions: $u(0, t) = 0$ \hspace{1cm} $u(L, t) = 0$

Initial Conditions: $u(x, 0) = f(x)$
Derivation

\[ \frac{\partial u}{\partial t} = X'T' \]

\[ \frac{\partial u}{\partial x} = X'T \]

\[ \frac{\partial^2 u}{\partial x^2} = X'''T \]

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad XT' = kX'''T \]

The only way this can occur is if both sides equal a constant

\[ \frac{T'}{kT} = \frac{X''}{X} = -\lambda \]
Now Solve for $X$

$X'' = -\lambda X \quad \Rightarrow \quad X'' + \lambda X = 0$

$(D^2 + \lambda)X = 0 \quad \Rightarrow \quad D = \pm \sqrt{\lambda} i$

$\Rightarrow X = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

Using the boundary conditions we get:

$X(0) = C_1 = 0 \quad \text{because } \cos(0) = 1 \text{ and } \sin(0) = 0$

$X(L) = C_2 \sin(\sqrt{\lambda}L) = 0 \quad \Rightarrow \quad \sin(\sqrt{\lambda}L) = 0 \quad \text{when } \sqrt{\lambda}L = n\pi \quad \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n \in \mathbb{Z}$

Therefore: $X = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right)$ by the Superposition Principle

Now Solve for $T$

$T' = -\lambda k T \quad \Rightarrow \quad T' + \lambda k T = 0$

$(D + \lambda k)T = 0 \quad \Rightarrow \quad D = -\lambda k$

$\Rightarrow \quad T = Be^{-\lambda k t}$

Thus, we get: $u(x, t) = X(x)T(t)$

$u(x, t) = Be^{-\lambda k t} \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n} x) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n} x) e^{-\lambda_n k t}$
Using the initial condition: \( u(x,0) = f(x) \)

\[
u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n} x) = f(x)
\]

\[
\Rightarrow A_n = \frac{1}{L} \int_{0}^{L} f(x) \sin(\sqrt{\lambda_n} x) dx
\]

Final Solution

\[
u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n} x)e^{-\lambda_n kt} \text{ where } A_n = \frac{1}{L} \int_{0}^{L} f(x) \sin(\sqrt{\lambda_n} x) dx
\]
Unfiltered Heat Equation
Filtered Heat Equation
Euler and d’Alembert

- They too solved the wave equation, but by using “arbitrary functions representing two waves, one moving along the string to the right, the other to the left, with a velocity equal to the constant $c$."
- Bernoulli’s solution was a summation of sine waves
- No one could tell if the two solutions could be reconciled
“Fourier showed that almost any function, when regarded as a periodic function over a given interval, can be represented by a trigonometric series of the form

\[ f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x \ldots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \ldots \]

where the coefficients \( a_i \) and \( b_i \) can be found from \( f(x) \) by computing certain integrals.”
Modern Application: Schrödinger

- The Schrödinger Equation was published in a series of papers on wave mechanics in 1926.
- The solutions to the Schrödinger Equation are wave functions that express the probability that a particle (specifically the electron of a hydrogen atom) will be in a certain location at any observation.
Solution to Schrödinger

- As with the heat and wave equations, we assume that the Schrödinger Equation is separable:
  
  \[ u(x,t) = X(x)T(t) \]

The Schrödinger Equation

\[ \frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial t^2} \]

Boundary Conditions: \[ \psi(0,t) = 0 \quad \psi(L,t) = 0 \]

Initial Conditions: \[ \psi(x,0) = f(x) \]

\[ \hbar = 1.054 \times 10^{-27} \text{ erg s} \]
Solution

Assume $\Psi(x,t)$ is separable

$$\psi(x,t) = X(x)T(t)$$

$$\frac{\partial \psi}{\partial t} = XT'$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{i\hbar}{2m} X''$$

Let $s = \frac{\hbar}{2m}$

$$\frac{\partial \psi}{\partial t} = is \frac{\partial^2 \psi}{\partial x^2}$$

$$XT' = isX''T$$

By separation of variables we get:

$$\frac{T'}{isT} = \frac{X''}{X} = -c$$

Similar to the Heat and Wave Equations
Solve for $T$:

To solve for $T$:

\[ \frac{T'}{T} = isc \]

\[ \ln(T) = -isc + d \]

\[ T = e^{-isc+d} = e^{-isc}e^d \]

\[ T = be^{-isc} \]

Solve for $X$:

To solve for $X$:

\[ \frac{X''}{X} = -c \]

\[ X'' = -cX \Rightarrow X'' + cX = 0 \]

\[ (D^2 + c)X = 0 \Rightarrow D = \pm \sqrt{ci} \]

\[ X = a_1 \cos(\sqrt{c}x) + a_2 \sin(\sqrt{c}x) \]
Using the boundary conditions we get the same results as before:

\[ X = a_2 \sin \sqrt{c}L \quad \text{Where } a_1 = 0 \quad \text{and } c_n = \left( \frac{n\pi}{L} \right)^2 \quad n \in \mathbb{Z} \]

Therefore we get \( \psi(x, t) = X(x)T(t) \)

\[ \psi(x, t) = a_2 \sin(\sqrt{c}L) e^{-i\omega t} \]

By the superposition principle

\[ \psi(x, t) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{L} x \right) e^{-i\left( \frac{n\pi}{L} \right)^2 t} \]

Using the initial conditions:

\[ \psi(x, 0) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{L} x \right) = f(x) \]

\[ \Rightarrow b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi}{L} x \right) dx \]
\[ \psi(x, t) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{L} x \right) e^{i\left( \frac{n\pi}{L} \right)^2 t} \]
Where 
\[ b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi}{L} x \right) dx \]
The graphs were produced using the open source program SAGE by Prof. W. David Joyner, USNA

http://modular.math.washington.edu/sage
References