Macromechanical Analysis of a Lamina
Generalized Hooke’s Law

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{ij} \]

\[ C_{ijkl} \] is a 9 x 9 matrix!
Hooke’s Law

- Assume
  - linear elastic behavior
  - small deformations

\[ \sigma = E\varepsilon \]

Uniaxial loading
Similarly for the deformations in the \( y \)- and \( z \)-directions

\[
\varepsilon_y = \frac{1}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right]
\]

\[
\varepsilon_z = \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right]
\]
Triaxial Stress State

Strain-Stress Relations

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu(\sigma_y + \sigma_z) \right] \]
\[ \varepsilon_y = \frac{1}{E} \left[ \sigma_y - \nu(\sigma_x + \sigma_z) \right] \]
\[ \varepsilon_z = \frac{1}{E} \left[ \sigma_z - \nu(\sigma_x + \sigma_y) \right] \]
Triaxial Stress State

Stress-Strain Relations

\[
\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z) \right]
\]

\[
\sigma_y = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z) \right]
\]

\[
\sigma_z = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y) \right]
\]
Similarly for shear stresses
Shear Stress Strain Relationships

- Shear stresses are independent of each other and all axial stresses
- Shear strains are independent of each other and all axial strains
- Each obeys a simple linear elastic model

\[
\tau = G\gamma
\]

\[
G = \frac{E}{2(1 + \nu)}
\]

\(G\) - Shear Modulus

© 2003, P. Joyce
Shear Stress-Strain Relationships

\[
\begin{align*}
\tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \\
\tau_{xz} &= G\gamma_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz} \\
\tau_{yz} &= G\gamma_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}
\end{align*}
\]
### Generalized Hooke’s Law

The strain-stress relations for an isotropic material can be expressed in matrix form as:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{zx} \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{zx} \\
\tau_{xy}
\end{bmatrix}
\]

This matrix is known as the compliance matrix.

Strain-stress relations for an isotropic material/matrix form.

© 2003, P. Joyce
### Generalized Hooke’s Law

Stress-strain relations for an isotropic material /matrix form

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{zx} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
E(1−\nu)
& \nu E
& \nu E
& 0 \\
(1−2\nu)(1+\nu)
& (1−2\nu)(1+\nu)
& (1−2\nu)(1+\nu)
& 0 \\
\nu E
& E(1−\nu)
& \nu E
& 0 \\
(1−2\nu)(1+\nu)
& (1−2\nu)(1+\nu)
& (1−2\nu)(1+\nu)
& 0 \\
\nu E
& \nu E
& E(1−\nu)
& 0 \\
(1−2\nu)(1+\nu)
& (1−2\nu)(1+\nu)
& (1−2\nu)(1+\nu)
& 0 \\
0
& 0
& 0
& G \\
0
& 0
& 0
& 0 \\
0
& 0
& 0
& G \\
0
& 0
& 0
& G
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{zx} \\
\gamma_{xy}
\end{bmatrix}
\]
<table>
<thead>
<tr>
<th>Material</th>
<th>No. of independent elastic constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>General anisotropic material</td>
<td>81</td>
</tr>
<tr>
<td>Anisotropic material (considering symmetry of stress and strain tensors)</td>
<td>36</td>
</tr>
<tr>
<td>Anisotropic material with energy considerations</td>
<td>21</td>
</tr>
<tr>
<td>General orthotropic material</td>
<td>9</td>
</tr>
<tr>
<td>Transversely orthotropic material</td>
<td>5</td>
</tr>
<tr>
<td>Isotropic material</td>
<td>2</td>
</tr>
</tbody>
</table>
Stress-Strain Relations in Composite Materials

- **Orthotropic material** ~ has three mutually perpendicular plans of material symmetry.
- **Specially orthotropic material** ~ when the reference system of coordinates is selected along principal planes of material symmetry.
- **Transversely isotropic material** ~ one of its principal planes is a plane of isotropy (properties are the same in all directions.)
Thanks to symmetry of the stress and strain tensors, the compliance matrix reduces to a 6 x 6 matrix, introduce a contracted notation.

\[
\begin{align*}
\sigma_{11} &= \sigma_1, \sigma_{22} = \sigma_2, \sigma_{33} = \sigma_3, \\
\sigma_{23} &= \sigma_4, \sigma_{31} = \sigma_5, \sigma_{12} = \sigma_6 \\
\varepsilon_{11} &= \varepsilon_1, \varepsilon_{22} = \varepsilon_2, \varepsilon_{33} = \varepsilon_1, \\
2\varepsilon_{23} &= \varepsilon_4, 2\varepsilon_{31} = \varepsilon_5, 2\varepsilon_{12} = \varepsilon_6 \\
C_{1111} &= C_{11}, C_{1122} = C_{12}, C_{1133} = C_{13}, C_{1123} = 2C_{14}, C_{1131} = 2C_{15}, C_{1112} = 2C_{16} \\
C_{2211} &= C_{21}, C_{2222} = C_{22}, C_{2233} = C_{23}, C_{2223} = 2C_{24}, C_{2231} = 2C_{25}, C_{2212} = 2C_{26}
\end{align*}
\]
Stress-Strain Relations for Thin UD Lamina

Assumed to be under a state of plane stress

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_6
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_6
\end{bmatrix}
\]

Fully characterized by 4 independent constants, $Q_{ij} \sim$ reduced stiffnesses
Reduced Stiffness Matrix?

- If the stiffness matrix is the inverse of the compliance matrix, what is the reduced stiffness matrix?

\[ Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}} \quad (i, j = 1, 2, 6) \]
Relations Between Mathematical and Engineering Constants

\[ Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \]
\[ Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \]
\[ Q_{12} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} \]
\[ Q_{66} = G_{12} \]
You Said Four Independent Constants?

- From symmetry of the compliance matrix:

\[
\begin{align*}
\sigma_{ij} & = \sigma_{ji} \\
E_i & = E_j
\end{align*}
\]

- The above can also be deduced from Betti’s reciprocal law according to which transverse deformation due to a stress applied in the longitudinal direction is equal to the longitudinal deformation due to an equal stress applied in the transverse direction.
Stress-Strain Relations

Also expressed in terms of compliances

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_6
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{12} & S_{22} & 0 \\
0 & 0 & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\varepsilon_6
\end{bmatrix}
\]
Relations between Mathematical and Engineering Constants

\[ S_{11} = \frac{1}{E_1} \]
\[ S_{22} = \frac{1}{E_2} \]
\[ S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2} \]
\[ S_{66} = \frac{1}{G_{12}} \]

That’s Better!
For a graphite/epoxy UD laminate, find the following:

1.) Compliance matrix
2.) Minor Poisson’s ratio
3.) Reduced Stiffness Matrix
4.) Strains in the 1-2 coordinate system if the applied stresses are

\[ \sigma_1 = 2 \text{ MPa}, \sigma_2 = -3 \text{ MPa}, \tau_{12} = 4 \text{ MPa} \]
# Sample Data

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Units</th>
<th>Glass/epoxy</th>
<th>Boron/epoxy</th>
<th>Graphite/epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber volume fraction</td>
<td>$V_f$</td>
<td>-</td>
<td>0.45</td>
<td>0.50</td>
<td>0.70</td>
</tr>
<tr>
<td>Long. elastic modulus</td>
<td>$E_1$</td>
<td>GPa</td>
<td>38.6</td>
<td>204</td>
<td>181</td>
</tr>
<tr>
<td>Trans. elastic modulus</td>
<td>$E_2$</td>
<td>GPa</td>
<td>8.27</td>
<td>18.50</td>
<td>10.30</td>
</tr>
<tr>
<td>Major Poisson’s ratio</td>
<td>$\nu_{12}$</td>
<td>-</td>
<td>0.26</td>
<td>0.23</td>
<td>0.28</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>$G_{12}$</td>
<td>GPa</td>
<td>4.14</td>
<td>5.59</td>
<td>7.17</td>
</tr>
</tbody>
</table>
Sample Calculation

The compliance matrix elements are calculated as follows:

\[ S_{11} = \frac{1}{E_1} = \frac{1}{181(10^9)} = 0.5525(10^{-11}) \]

\[ S_{12} = \frac{\nu_{12}}{E_1} = \frac{0.28}{181(10^9)} = -0.1547(10^{-11}) \]

\[ S_{22} = \frac{1}{E_2} = \frac{1}{10.3(10^9)} = 0.9709(10^{-10}) \]

\[ S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17(10^9)} = 0.1395(10^{-9}) \]

(all terms have units of Pa\(^{-1}\))
From Betti’s reciprocal law:

\[
\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}
\]

\[
\nu_{21} = \frac{(0.28)}{181(10^9)} (10.3)(10^9) = 0.01593
\]
The stiffness matrix elements are calculated as follows:

\[ Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} = \frac{181(10^9)}{1 - (0.28)(0.01593)} = 181.8(10^9) \]

\[ Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} = \frac{(0.28)(10.3)(10^9)}{1 - (0.28)(0.01593)} = 2.897(10^9) \]

\[ Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} = \frac{10.3(10^9)}{1 - (0.28)(0.01593)} = 10.35(10^9) \]

\[ Q_{66} = G_{12} = 7.17(10^9) \]

(all terms have units of Pa)
Sample Calculation

The stiffness matrix can also be calculated by inverting the compliance matrix of Part 1:

\[
[\mathcal{Q}] = \begin{bmatrix}
0.5525(10^{-11}) & -0.1547(10^{-11}) & 0 \\
-0.1547(10^{-11}) & 0.9709(10^{-10}) & 0 \\
0 & 0 & 0.1395(10^{-9})
\end{bmatrix}^{-1}
\]

\[
[\mathcal{Q}] = \begin{bmatrix}
181.8(10^9) & 2.897(10^9) & 0 \\
2.897(10^9) & 10.35(10^9) & 0 \\
0 & 0 & 7.17(10^9)
\end{bmatrix}
\] (all terms have units of Pa)
The strains in the 1-2 coordinate system are calculated as follows:

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{12} & S_{22} & 0 \\
0 & 0 & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
2(10^{-6}) \\
-3(10^{-6}) \\
4(10^{-6})
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
15.69 \\
-294.4 \\
557.9
\end{bmatrix} 
(10^{-6} \text{ is microstrain})
\]
Stress-Strain Relations for Thin Angle Lamina

Generally, a laminate does not consist only of UD laminae because of their stiffness and strength properties in the transverse direction.

Hence, in most laminates, some laminae are placed at an angle.
Stress-Strain Relations for Thin Angle Lamina

- The axes in the $x$-$y$ coordinate system are called the global axes of the off-axes.
- The axes in the 1-2 coordinate system are called the material axes or the local axes, where direction 1 is parallel to the fibers (also called the longitudinal direction) and direction 2 is perpendicular to the fibers (also called the transverse direction.)
- The angle between the two axes is denoted by the angle $\theta$. 

© 2003, P. Joyce
Stress-Strain Relations for Thin Angle Lamina

- The stress-strain relationship in the 1-2 coordinate system has already been established.
- From Mechanics of Materials, the stresses in the global and material axes are related to each other through the angle of the lamina, \( \theta \).

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
= [T(\theta)]^{-1}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

Where \([T(\theta)]\) is called the transformation matrix and is defined as

\[
[T(\theta)] = \begin{bmatrix}
c^2 & s^2 & 2sc \\
s^2 & c^2 & -2sc \\
-sc & sc & c^2 - s^2
\end{bmatrix}
\]

thus \([T(\theta)]^{-1} = \begin{bmatrix}
c^2 & s^2 & -2sc \\
s^2 & c^2 & 2sc \\
-sc & -sc & c^2 - s^2
\end{bmatrix} = [T(-\theta)]\]
Stress-Strain Relations for Thin Angle Lamina

Using the stress-strain equation in the material axes together with the transformation equation we obtain:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = [T]^{-1} \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
\]
Stress-Strain Relations for Thin Angle Lamina

- Similarly, the strains in the global and material coordinate axes are related through the transformation matrix

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\frac{1}{2} \gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
[T]
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\frac{1}{2} \gamma_{xy}
\end{bmatrix}
\]

which can be rewritten as

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\frac{1}{2} \gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
[R][T][R]^{-1}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\frac{1}{2} \gamma_{xy}
\end{bmatrix}
\]

where \([R]\) is the Reuter matrix and is defined as

\[
[R] =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]
Stress-Strain Relations for Thin Angle Lamina

- Multiplying out the first five matrices on the RHS of the previous equation we obtain the transformed reduced stiffness matrix, \([Q_{xy}]\).
- Thus, \([\sigma]_{x,y} = [Q]_{x,y} [\varepsilon]_{x,y}\).
- Summarizing,

\[
\begin{align*}
Q_{xx} &= e^4 Q_{11} + s^4 Q_{22} + 2e^2 s^2 Q_{12} + 4e^2 s^2 Q_{66} \\
Q_{yy} &= s^4 Q_{11} + c^4 Q_{22} + 2c^2 s^2 Q_{12} + 4c^2 s^2 Q_{66} \\
Q_{xy} &= c^2 s^2 Q_{11} + c^2 s^2 Q_{22} + (c^4 + s^4) Q_{12} - 4c^2 s^2 Q_{66} \\
Q_{xs} &= c^3 s Q_{11} + c^3 s Q_{22} + (c s^3 - c^3 s) Q_{12} + 2(c s^3 - c^3 s) Q_{66} \\
Q_{ys} &= c^3 s Q_{11} - c^3 s Q_{22} + (c^3 s - c s^3) Q_{12} + 2(c^3 s - c s^3) Q_{66} \\
Q_{ss} &= c^2 s^2 Q_{11} + c^2 s^2 Q_{22} - 2c^2 s^2 Q_{12} + (c^2 - s^2)^2 Q_{66}
\end{align*}
\]

- The subscript s corresponds to shear stress or strain components in the x-y system, i.e., \(\tau_s = \tau_{xy}\) and \(\gamma_s = \gamma_{xy}\).
Strain-Stress Relations for Thin Angle Lamina

Similarly, the transformed compliance can be obtained:

\[
\begin{align*}
S_{xx} &= c^4 S_{11} + s^4 S_{22} + 2c^2 s^2 S_{12} + 4c^2 s^2 S_{66} \\
S_{yy} &= s^4 S_{11} + c^4 S_{22} + 2c^2 s^2 S_{12} + 4c^2 s^2 S_{66} \\
S_{xy} &= c^2 s^2 S_{11} + c^2 s^2 S_{22} + (c^4 + s^4) S_{12} - 4c^2 s^2 S_{66} \\
S_{xs} &= 2c^3 s S_{11} - 2c s^3 S_{22} + 2(c s^3 - c^3 s) S_{12} + (c s^3 - c^3 s) S_{66} \\
S_{ys} &= 2c s^3 S_{11} - 2c^3 s S_{22} + 2(c^3 s - c s^3) S_{12} + (c^3 s - c s^3) S_{66} \\
S_{ss} &= 4c^2 s^2 S_{11} + 4c^2 s^2 S_{22} - 8c^2 s^2 S_{12} + (c^2 - s^2)^2 S_{66}
\end{align*}
\]

Thus, \([\varepsilon]_{x,y} = [S]_{x,y} [\sigma]_{x,y}\)
Strain-Stress Relations for Thin Angle Lamina

- How about in terms of Engineering Constants?
- If we imagine a series of simple experiments on an element with sides parallel to the $x$- and $y$-axes, we obtain:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_s
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{E_x} & \frac{\nu_{xy}}{E_y} & \frac{\eta_{xx}}{G_{xy}} \\
\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{yy}}{G_{xy}} \\
\frac{\eta_{xs}}{E_x} & \frac{\eta_{ys}}{E_y} & \frac{1}{G_{xy}}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_s
\end{bmatrix}
\]
Strain-Stress Relations for Thin Angle Lamina

- What is $\eta$? → shear coupling coefficient
- $\eta_{xs}$, the first subscript denotes normal loading in the x-direction; the second subscript denotes shear strain.

\[
\begin{align*}
\eta_{xs} &= \gamma_s / \varepsilon_x \\
\eta_{ys} &= \gamma_s / \varepsilon_y \\
\eta_{sx} &= \varepsilon_x / \gamma_s \\
\eta_{sy} &= \varepsilon_y / \gamma_s
\end{align*}
\]
Strain-Stress Relations for Thin Angle Lamina

- Comparison of equivalent strain-stress relations yields the following relationships:

<table>
<thead>
<tr>
<th>Term</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{xx}$</td>
<td>$\frac{1}{E_x}$</td>
</tr>
<tr>
<td>$S_{yy}$</td>
<td>$\frac{1}{E_y}$</td>
</tr>
<tr>
<td>$S_{ss}$</td>
<td>$\frac{1}{G_{xy}}$</td>
</tr>
<tr>
<td>$S_{xy}$</td>
<td>$S_{yx} = \frac{\nu_{xy}}{E_x} = \frac{-\nu_{yx}}{E_y}$</td>
</tr>
<tr>
<td>$S_{xx}$</td>
<td>$S_{xx} = \frac{\eta_{xx}}{E_x} = \frac{\eta_{sx}}{G_{xy}}$</td>
</tr>
<tr>
<td>$S_{yy}$</td>
<td>$S_{yy} = \frac{\eta_{yy}}{E_y} = \frac{\eta_{sy}}{G_{xy}}$</td>
</tr>
<tr>
<td>$E_x$</td>
<td>$\frac{1}{S_{xx}}$</td>
</tr>
<tr>
<td>$E_y$</td>
<td>$\frac{1}{S_{yy}}$</td>
</tr>
<tr>
<td>$G_{xy}$</td>
<td>$\frac{1}{S_{ss}}$</td>
</tr>
<tr>
<td>$\nu_{xy}$</td>
<td>$\frac{-S_{xy}}{S_{xx}}$</td>
</tr>
<tr>
<td>$\nu_{yx}$</td>
<td>$\frac{S_{yx}}{S_{yy}}$</td>
</tr>
<tr>
<td>$\eta_{xx}$</td>
<td>$\frac{S_{ss}}{S_{xx}}$</td>
</tr>
<tr>
<td>$\eta_{yy}$</td>
<td>$\frac{S_{ss}}{S_{yy}}$</td>
</tr>
<tr>
<td>$\eta_{xy}$</td>
<td>$\frac{S_{xy}}{S_{yy}}$</td>
</tr>
<tr>
<td>$\eta_{yx}$</td>
<td>$\frac{S_{yx}}{S_{xx}}$</td>
</tr>
</tbody>
</table>

© 2003, P. Joyce
Sample Calculation

- Find the following for a 60° angle lamina of graphite/epoxy.
  - Transformed compliance matrix
  - Transformed reduced stiffness matrix
  - Global strains
  - Local strains
  - If the applied stresses are $\sigma_x = 2$ MPa, $\sigma_y = -3$ MPa, $\tau_{xy} = 4$ MPa
Sample Calculation

From the previous example:

\[ S_{11} = 0.5525 \times 10^{-11} \]
\[ S_{12} = -0.1547 \times 10^{-11} \]
\[ S_{22} = 0.9709 \times 10^{-10} \]
\[ S_{66} = 0.1395 \times 10^{-9} \]

(all terms have units of Pa\(^{-1}\))
Sample Calculation

The transformed compliance matrix elements are calculated as follows:

\[
S_{xx} = c^4 S_{11} + s^4 S_{22} + 2 c^2 s^2 S_{12} + 4 c^2 s^2 S_{66} = 0.8053(10^{-10})
\]
\[
S_{yy} = s^4 S_{11} + c^4 S_{22} + 2 c^2 s^2 S_{12} + 4 c^2 s^2 S_{66} = -0.7878(10^{-11})
\]
\[
S_{xy} = c^2 s^2 S_{11} + c^2 s^2 S_{22} + (c^4 + s^4) S_{12} - 4 c^2 s^2 S_{66} = -0.3234(10^{-10})
\]
\[
S_{xs} = 2 c^3 s S_{11} - 2 c s^3 S_{22} + 2 (c s^3 - c^3 s) S_{12} + (c s^3 - c^3 s) S_{66} = 0.3475(10^{-10})
\]
\[
S_{ys} = 2 c s^3 S_{11} - 2 c^3 s S_{22} + 2 (c^3 s - c s^3) S_{12} + (c^3 s - c s^3) S_{66} = -0.4696(10^{-10})
\]
\[
S_{ss} = 4 c^2 s^2 S_{11} + 4 c^2 s^2 S_{22} - 8 c^2 s^2 S_{12} + (c^2 - s^2)^2 S_{66} = 0.1141(10^{-9})
\]

(All terms are in Pa⁻¹)
Next, invert the transformed compliance matrix $[S]$ to obtain the transformed reduced stiffness matrix $[Q]$.

\[
[Q] = [S]^{-1} = \begin{bmatrix}
0.8053(10^{-10}) & -0.7878(10^{-10}) & -0.3234(10^{-10}) \\
-0.7878(10^{-10}) & 0.3475(10^{-10}) & -0.4696(10^{-10}) \\
-0.3234(10^{-10}) & -0.4696(10^{-10}) & 0.1141(10^{-9})
\end{bmatrix}
\]

\[
[Q] = \begin{bmatrix}
0.2365 & 0.3246 & 0.2005 \\
0.3246 & 1.094 & 0.5419 (10^{11}) \\
0.2005 & 0.5419 & 0.3674
\end{bmatrix}
\]

(all terms in Pa)
The global strains in the x-y plane are given by 

\[ \varepsilon_{x,y} = S_{x,y} \sigma_{x,y} \]

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
= \begin{bmatrix}
0.8053(10^{-10}) & -0.7878(10^{-10}) & -0.3234(10^{-10}) \\
-0.7878(10^{-10}) & 0.3475(10^{-10}) & -0.4696(10^{-10}) \\
-0.3234(10^{-10}) & -0.4696(10^{-10}) & 0.1141(10^{-9})
\end{bmatrix}
\begin{bmatrix}
2 \\
-3 \times 10^6 \\
4
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
= \begin{bmatrix}
0.5534(10^{-4}) \\
0.5328(10^{-3})
\end{bmatrix}
\]
Sample Calculation

The local strains in the lamina can be calculated using the Transformation equation.

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\frac{1}{2} \gamma_{12}
\end{bmatrix}
= [T]
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\frac{1}{2} \gamma_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\frac{1}{2} \gamma_{12}
\end{bmatrix}
= \begin{bmatrix}
\cos^2 60 & \sin^2 60 & 2 \cos 60 \sin 60 & 0.5534 \times 10^{-4} \\
\sin^2 60 & \cos^2 60 & -2 \cos 60 \sin 60 & -0.3078 \times 10^{-3} \\
-\cos 60 \sin 60 & \cos 60 \sin 60 & \cos^2 60 - \sin^2 60 & 0.5328 \times 10^{-3} / 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
= \begin{bmatrix}
0.1367 \times 10^{-4} \\
-0.2662 \times 10^{-3} \\
-0.5809 \times 10^{-3}
\end{bmatrix}
\]

© 2003, P. Joyce
Transformation of Engineering Constants

Flow chart for determination of transformed elastic constants of UD lamina.
Macromechanical Strength Parameters

- From a macromechanical POV, the strength of a lamina is an anisotropic property.
- It is desirable, for example, to correlate the strength along an arbitrary direction to some basic strength parameters (analogous to micromechanic definitions before.)
Various theories have been developed for studying the failure of an angle lamina.

Generally based on the normal and shear strengths of a UD lamina.

Need to consider tension and compression.

UD lamina has 2 material axes, 1-direction parallel to the fibers and 2-direction which is perpendicular to the fibers.

Hence there are 4 normal strength parameters for UD lamina.

- Tensile strength in fiber direction
- Transverse tensile strength
- Compressive strength in fiber direction
- Transverse compressive strength

The fifth strength parameter is the shear strength.

© 2003, P. Joyce
Strength Failure Theories of an Angle Lamina

- Unlike the stiffness parameters, these strength parameters cannot be transformed directly for an angle lamina.
- Hence, the failure theories are based on first finding the stresses in the material axes and then using these five strength parameters of a UD lamina to find whether the lamina has failed.
Macromechanical Strength Parameters

- Also predict transverse compressive strength and in-plane shear strength using micromechanics.
- Failure mechanisms vary greatly with material properties and type of loading.
- Even when predictions are accurate with regard to failure initiation at critical points, they are only approximate as far as global failure of the lamina is concerned.
- Furthermore, the possible interaction of failure mechanisms makes it difficult to obtain reliable strength predictions under a general type of loading.
- A macromechanical or phenomenological approach to failure analysis may be preferable.
Macromechanical Strength Parameters

- This characterization recognizes the fact that most composite materials have different strengths in tension and compression.

- By convention the sign of the shear stress is immaterial, as long as the shear strength is referred to the principal material directions.

- Exception, refers to the case when the shear stress is applied at an angle wrt the principal material directions.

- Since most composites have different tensile and compressive strengths and they are weakest in transverse tension, it follows that in this case the lamina would be stronger under positive shear.
Macromechanical Strength Parameters

\[ \sigma_y = \tau_6 \]

\[ \sigma_x = -\tau_6 \]

Positive shear stress

Shear stress acting along principal material axes.

© 2003, P. Joyce
Macromechanical Strength Parameters

\[ \sigma_y = \tau_6 \]

\[ \sigma_x = -\tau_6 \]

Negative shear stress

Shear stress acting along principal material axes.

© 2003, P. Joyce
Macromechanical Strength
Parameters

Positive shear stress

Shear stress acting at 45° wrt principal material axes.
Macromechanical Strength Parameters

Shear stress acting at 45° wrt principal material axes.

\[ \sigma_1 = -\tau_s \]

\[ \sigma_2 = \tau_s \]

Negative shear stress
Macromechanical Failure Theories

- Given a state of stress, the principal stresses and their directions are obtained by stress transformation (independent of material properties.)
- The principal strains and their directions are obtained by using the appropriate anisotropic stress-strain relations and strain transformation.
- In general, the principal stress, principal strain, and material symmetry directions do not coincide.
- Since strength varies with orientation, maximum stress alone is not the critical factor in failure.
Macromechanical Failure Theories

- An anisotropic failure theory is needed.
- Failure criteria for homogeneous isotropic materials, such as
  - Maximum normal stress (Rankine),
  - Maximum shear stress (Tresca),
  - Maximum distortional energy (von Mises),
  - and so forth are well established.
- More than 40 anisotropic theories have been proposed – look at the four most widely used.
Maximum Stress Failure Theory

- Related to the Maximum Normal stress theory by Rankine and the Maximum Shear stress theory by Tresca.
- The stresses acting on a lamina are resolved into the normal and shear stresses in the material axes.
- Failure is predicted in a lamina, if any of the normal or shear stresses in the material axes are equal to or greater than the corresponding ultimate strengths of a UD lamina.

\[
-\left(\sigma^C_1\right)_{\text{ult}} < \sigma_1 < \left(\sigma^T_1\right)_{\text{ult}}, \quad -\left(\sigma^C_2\right)_{\text{ult}} < \sigma_2 < \left(\sigma^T_2\right)_{\text{ult}}, \quad -\left(\tau_{12}\right)_{\text{ult}} < \tau_{12} < \left(\tau_{12}\right)_{\text{ult}}
\]

- Each component of stress is compared with the corresponding strength and hence does not have an interaction with the others.
Sample Calculation

- Find the off-axis shear strength of a 60° graphite/epoxy lamina using the Maximum Stress failure criteria.
- Assume the following stress state

\[
\sigma_x = 0, \sigma_y = 0, \tau_{xy} = \tau,
\]

- Find the stresses along the principal material axes, using the Transformation Equation.

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
0.2500 & 0.7500 & 0.8660 \\
0.7500 & 0.2500 & -0.8660 \\
-0.4330 & 0.4330 & -0.5000
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\tau
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
0.866\tau \\
-0.866\tau \\
-0.500\tau
\end{bmatrix}
\]
Sample Calculation

Applying the Maximum Stress Failure Criteria together with strength data for graphite/epoxy composites from the Data Sheet, we have

\[-1500 < 0.866 \tau < 1500\]
\[-246 < -0.866 \tau < 40\]
\[-68 < 0.500 \tau < 68\]

or

\[-1732 < \tau < 1732\]
\[-46.19 < \tau < 284.1\]
\[-136.0 < \tau < 136.0\]
The off-axis shear strength of a lamina is defined as the minimum of the positive and negative shear stress which can be applied to an angle lamina before failure.

Calculations show that $\tau_{xy} = 46.19$ MPa is the largest magnitude of shear stress one can apply to the 60° graphite/epoxy composite.

However, the largest positive shear stress one could apply is 136.0 MPa, and the largest negative shear stress one could apply is $-46.19$ MPa.

This shows that the maximum magnitude of allowable shear stress in other than the material axes direction depends on the sign of the shear stress.

This is because the tensile strength perpendicular to the fiber direction is much lower than the compressive strength perpendicular to the fiber direction.
A failure envelope is a 3D plot of the combinations of normal and shear stresses which can be applied to an angle lamina before failure.

Drawing 3D graphs is time consuming...

One may develop failure envelopes for constant shear stress, \( \tau_{xy} \), and then use the 2 normal stresses \( \sigma_x \) and \( \sigma_y \) as the 2 axes.

If the applied stress is within the failure envelope, the lamina is safe; otherwise it has failed.
Failure Envelopes

- For a UD lamina at a given shear stress loading, the failure envelope takes the form of a rectangle as shown.

- For a 60° lamina at a given shear stress loading, the failure envelope takes the form of a rectangle as shown.
Maximum Strain Failure Theory

- Based on the Maximum Normal Strain Theory by St. Venant and the Maximum Shear Stress Theory by Tresca.
- The strains applied to a lamina are resolved into the normal and shear stresses in the material axes.
- Failure is predicted in a lamina, if any of the normal or shear strains in the material axes are equal to or greater than the corresponding ultimate strains of a UD lamina.

\[-(\varepsilon^C_1)_{ult} < \varepsilon_1 < (\varepsilon^T_1)_{ult}, \quad -(\varepsilon^C_2)_{ult} < \varepsilon_2 < (\varepsilon^T_2)_{ult}, \quad -(\gamma_{12})_{ult} < \gamma_{12} < (\gamma_{12})_{ult}\]

- The ultimate strains can be found directly from the ultimate strength parameters and the elastic moduli, assuming the stress-strain response is linear until failure.
- Each component of strain is compared with the corresponding ultimate strain and hence does not have an interaction with the others.
- Yields different results from Maximum Stress Failure Theory, because the local strains in a lamina include the Poisson’s ratio effect (allows some interaction of stress components.)
Maximum Strain Failure Theory

Assume a general biaxial state of stress on an angle lamina.

Obtain the stress components along the principal material axes by stress transformation.

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = [T(\theta)]^{-1}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]
Maximum Strain Failure Theory

- Then the corresponding strain components can be calculated by means of the lamina stress-strain relations:

\[ \varepsilon_1 = \frac{\sigma_1}{E_1} - \nu_{21} \frac{\sigma_2}{E_2} \]
\[ \varepsilon_2 = \frac{\sigma_2}{E_2} - \nu_{12} \frac{\sigma_1}{E_1} \]
\[ \gamma_6 = \frac{\tau_6}{G_{12}} \]

- Next calculate the ultimate strains as follows

\[ (\varepsilon_1^T)^{ult} = (\frac{\sigma_1^T}{E_1})^{ult}, \quad (\varepsilon_1^C)^{ult} = (\frac{\sigma_1^C}{E_1})^{ult}, \quad (\varepsilon_2^T)^{ult} = (\frac{\sigma_2^T}{E_2})^{ult}, \quad (\varepsilon_2^C)^{ult} = (\frac{\sigma_2^C}{E_2})^{ult}, \quad (\gamma_{12})^{ult} = (\frac{\tau_{12}}{G_{12}})^{ult} \]
Failure subcriteria restated in terms of the stresses:

\[
\begin{align*}
\sigma_1 - \nu_{12} \sigma_2 &= \begin{cases} 
(\sigma_1^T)_{ult} & \text{when } \varepsilon_1 > 0 \\
-(\sigma_1^C)_{ult} & \text{when } \varepsilon_1 < 0
\end{cases} \\
\sigma_2 - \nu_{21} \sigma_1 &= \begin{cases} 
(\sigma_2^T)_{ult} & \text{when } \varepsilon_2 > 0 \\
-(\sigma_2^C)_{ult} & \text{when } \varepsilon_2 < 0
\end{cases} \\
-(\tau_{12})_{ult} &< \tau_{12} < (\tau_{12})_{ult}
\end{align*}
\]
Maximum Strain Failure Theory

- For a 2D state of stress with $\tau_6 = 0$, the failure envelope takes the form of a parallelogram with its center off the origin.
Tsai-Hill Failure Theory

- Based on the deviatoric or distortional energy failure theory of von Mises.
- Adapted to anisotropic materials by Hill.
- Then adapted to a UD lamina by Tsai.

\[
\left( \frac{\sigma_1}{(\sigma^T_1)_{ult}} \right)^2 - \left( \frac{\sigma_1 \sigma_2}{(\sigma^T_1)_{ult}} \right)^2 + \left( \frac{\sigma_2}{(\sigma^T_2)_{ult}} \right)^2 + \left( \frac{\tau_{12}}{(\tau_{12})_{ult}} \right)^2 < 1
\]

- Given the global stresses in a lamina, one can find the local stresses in a lamina and apply the above failure theory to determine whether or not the lamina has failed.
Tsai-Hill Failure Theory

- The failure envelope described by the Tsai-Hill criterion is a closed surface in the $\sigma_1, \sigma_2, \tau_{12}$ space.
- Failure envelopes for constant values of $k = \tau_{12}/(\tau_{12})_{ult}$ have the form

$$\frac{\sigma_1^2}{F_1^2} + \frac{\sigma_2^2}{F_2^2} - \frac{\sigma_1 \sigma_2}{F_1^2} = 1 - k^2$$

- Where:

$$F_1 = \begin{cases} \left(\sigma_{1T}^{ult}\right) & \text{when } \sigma_1 > 0 \\ \left(\sigma_{1C}^{ult}\right) & \text{when } \sigma_1 < 0 \end{cases}$$

$$F_2 = \begin{cases} \left(\sigma_{2T}^{ult}\right) & \text{when } \sigma_2 > 0 \\ \left(\sigma_{2C}^{ult}\right) & \text{when } \sigma_2 < 0 \end{cases}$$

- Graphically represents 4 different elliptical arcs joined at the $\sigma_1, \sigma_2$ axes.
Tsai-Hill Failure Theory

- Considers the interaction between the 3 UD lamina strength parameters, unlike the Maximum Stress and Maximum Strain Theories.
- Tsai-Hill Failure Theory is a Unified Theory and hence does not give the mode of failure like the Maximum Stress and Maximum Strain Theories.
Tsai-Wu Failure Theory

- Based on a general failure theory for anisotropic materials first proposed by Gol’denblat and Kopnov (1965).
- Capable of predicting strength under general states of stress for which no experimental data are available.
- Uses the concept of strength tensors.
- Has the form of an invariant formed from stress and strain tensor components.
- Has the capability to account for the difference between tensile and compressive strengths.
Tsai-Wu Failure Theory

Tsai and Wu (1971) proposed a modified tensor polynomial theory by assuming the existence of a failure surface in the stress space of the form —

\[ H_1 \sigma_1 + H_2 \sigma_2 + H_6 \tau_{12} + H_{11} \sigma_1^2 + H_{22} \sigma_2^2 + H_{66} \tau_{12}^2 + 2H_{12} \sigma_1 \sigma_2 < 1 \]

The coefficients are obtained by applying elementary loading conditions to the lamina. Thus —

\[
H_1 = \frac{1}{(\sigma_{11}^T)_{ult}} - \frac{1}{(\sigma_{11}^C)_{ult}} \quad H_2 = \frac{1}{(\sigma_{22}^T)_{ult}} - \frac{1}{(\sigma_{22}^C)_{ult}} \quad H_6 = 0 \\
H_{11} = \frac{1}{(\sigma_{11}^T)_{ult} (\sigma_{11}^C)_{ult}} \quad H_{22} = \frac{1}{(\sigma_{22}^T)_{ult} (\sigma_{22}^C)_{ult}} \quad H_{66} = \frac{1}{(\tau_{12})_{ult}^2}
\]

© 2003, P. Joyce
The remaining coefficient $H_{12}$ must be obtained by some type of biaxial testing.

Direct biaxial testing is not easy or practical to perform.

An easier test producing a biaxial state of stress is the off-axis tensile test.

For $\theta = 45^\circ$ we can measure the off-axis tensile strength, $\sigma$.

Then, $H_{12} = \frac{2}{\sigma} - \frac{(H_1 + H_2)}{\sigma} - \frac{1}{2} (H_{11} + H_{22} + H_{66})$.

Again produces an elliptical failure envelope.
Comparison with Experimental Results

![Graph showing comparison between Maximum Stress Failure Theory and Experimental Data Points](image)
Comparison with Experimental Results

![Graph showing comparison with experimental results](image-url)
Comparison with Experimental Results

- Tsai-Hill Failure Theory
- Experimental Data Points

Graph showing the relationship between stress ($\sigma_x$ in ksi) and angle of lamina ($\theta$ in degrees).
Comparison with Experimental Results

![Graph showing Comparison with Experimental Results](image)
Comparison with Experimental Results

Observations —

- The difference between the Maximum Stress and Maximum Strain Failure Theories and the experimental results is quite pronounced.
- Tsai-Hill and Tsai-Wu Failure Theories are in good agreement with experimentally obtained results.
- The cusps observed in the Maximum Stress and Maximum Strain Failure Theories correspond to the change in failure mode.
- The variation of the strength as a function of angle is smooth in the Tsai-Hill and Tsai-Wu Failure Theories.
## Macromechanical Failure Theories

<table>
<thead>
<tr>
<th>Theory</th>
<th>Physical Basis</th>
<th>Operational Convenience</th>
<th>Req’d experimental characterization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Stress</td>
<td>Tensile behavior of brittle material</td>
<td>Inconvenient</td>
<td>Few parameters by simple testing</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>No stress interaction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maximum Strain</td>
<td>Tensile behavior of brittle material</td>
<td>Inconvenient</td>
<td>Few parameters by simple testing</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Some stress interaction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deviatoric strain energy</td>
<td>Ductile behavior of anisotropic materials</td>
<td>Can be programmed</td>
<td>Biaxial testing is needed in addition to uniaxial testing</td>
</tr>
<tr>
<td>(Tsai-Hill)</td>
<td>“Curve fitting” for heterogeneous brittle composites</td>
<td>Different functions required for tensile and compressive strengths</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interactive tensor polynomial</td>
<td>Mathematically consistent</td>
<td>General and comprehensive; operationally simple</td>
<td>Numerous parameters</td>
</tr>
<tr>
<td>Tsai-Wu</td>
<td>Reliable “curve fitting”</td>
<td></td>
<td>Comprehensive experimental program needed.</td>
</tr>
</tbody>
</table>
References