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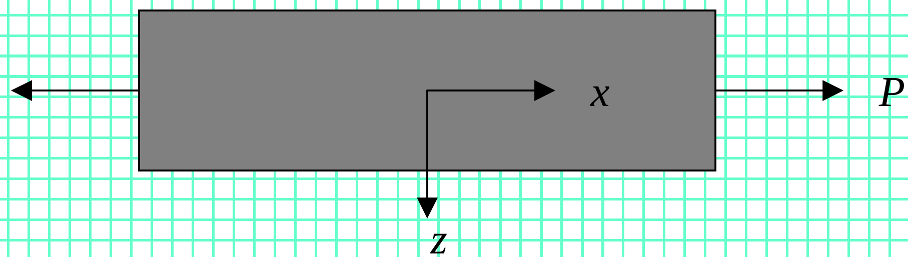
X-29 at High Angle of Attack

Macromechanical Analysis of Laminates

Stress –Strain Relations for an Isotropic Beam

Consider a prismatic beam of cross-section A under an applied axial load P.



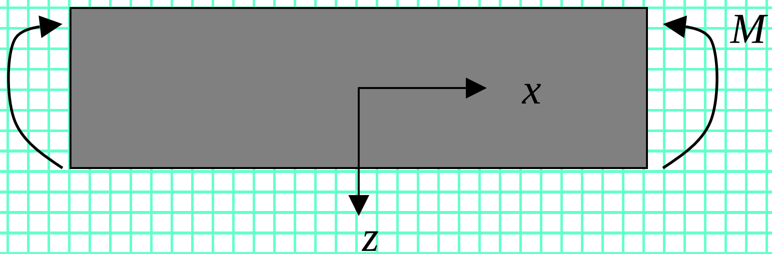
$$\sigma_{xx} = \frac{P}{A} \text{ and } \varepsilon_{xx} = \frac{P}{EA}$$

Assumes that the normal stress and strain are uniform and constant in the beam and are dependent on the load P being applied at the centroid of the cross-section.

One dimensional analysis

Stress –Strain Relations for an Isotropic Beam

Consider the same prismatic beam in a pure bending moment *M*.
The beam is assumed to initially straight and the applied loads pass through
 a plane of symmetry to avoid twisting.

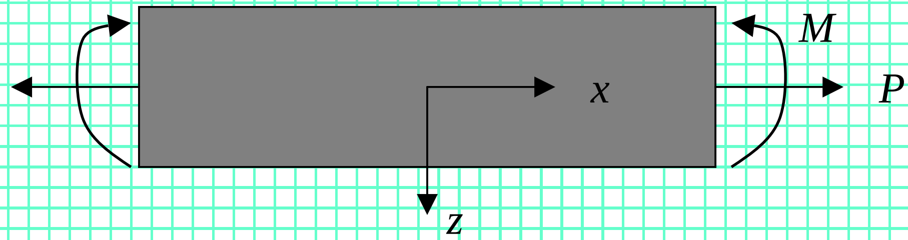


$$\varepsilon_{xx} = \frac{z}{\rho} \text{ and } \sigma_{xx} = \frac{Mz}{I}$$

Neglects transverse shear.
Assumes plane sections remain plane.
I s the second moment of area (often mistakenly referred to as the moment of inertia.)

Stress –Strain Relations for an Isotropic Beam

Finally consider the beam under combined loading.



$$\varepsilon_{xx} = \frac{P}{EA} + \frac{Mz}{EI}$$

$$\varepsilon_{xx} = \varepsilon_0 + z\left(\frac{1}{\rho}\right)$$

$$\varepsilon_{xx} = \varepsilon_0 + z\kappa$$

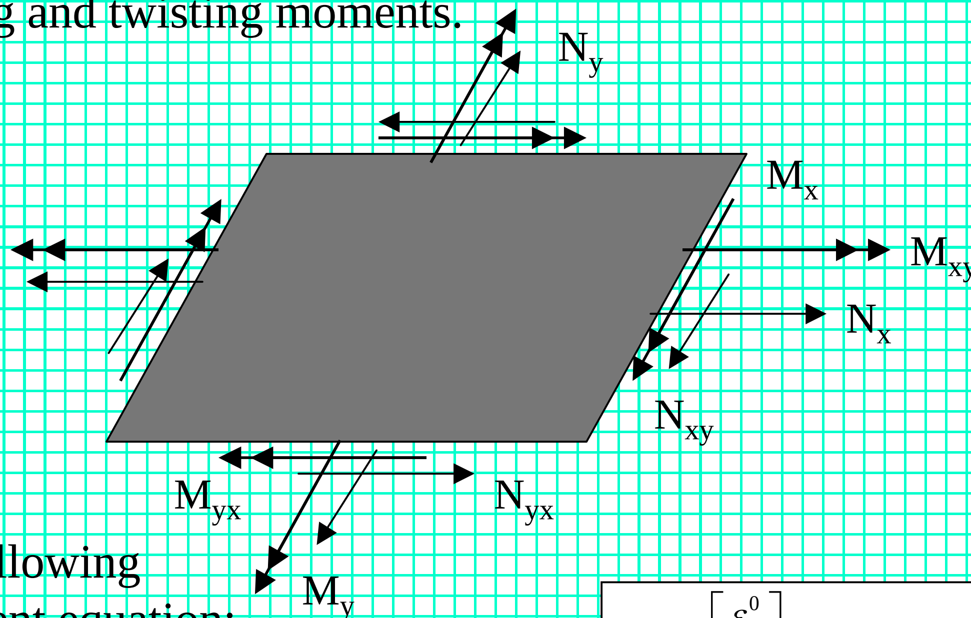
Where ε_0 is the strain at $y = 0$ (through the centroid), and κ = the curvature of the beam.

Strain-Displacement Equations for an Anisotropic Laminate

- Use Classical Lamination Theory (CLT) to develop similar relationships in 3D for a laminate (plate) under combined shear and axial forces and bending and twisting moments.
- The following assumptions are made to develop the relationships:
  - Each lamina is homogeneous and orthotropic
  - The laminate is thin and is loaded in plane only (plane stress)
  - Displacements are continuous and small throughout the laminate
  - Each lamina is elastic (stress-strain relations are linear)
  - No slip occurs between the lamina interfaces
  - Transverse shear strains are negligible
  - The transverse normal strain is negligible

Strain-Displacement Equations for an Anisotropic Laminate

Consider the general case of a plate under in-plane shear and axial loading, as well as bending and twisting moments.



Can derive the following Strain-displacement equation:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

where $\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix}$ are the midplane strains

and $\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$ are the midplane curvatures

Strain and Stress in a Laminate

If the strains are known at any point along the thickness of the laminate, the stress-strain equation calculates the global stresses in each lamina

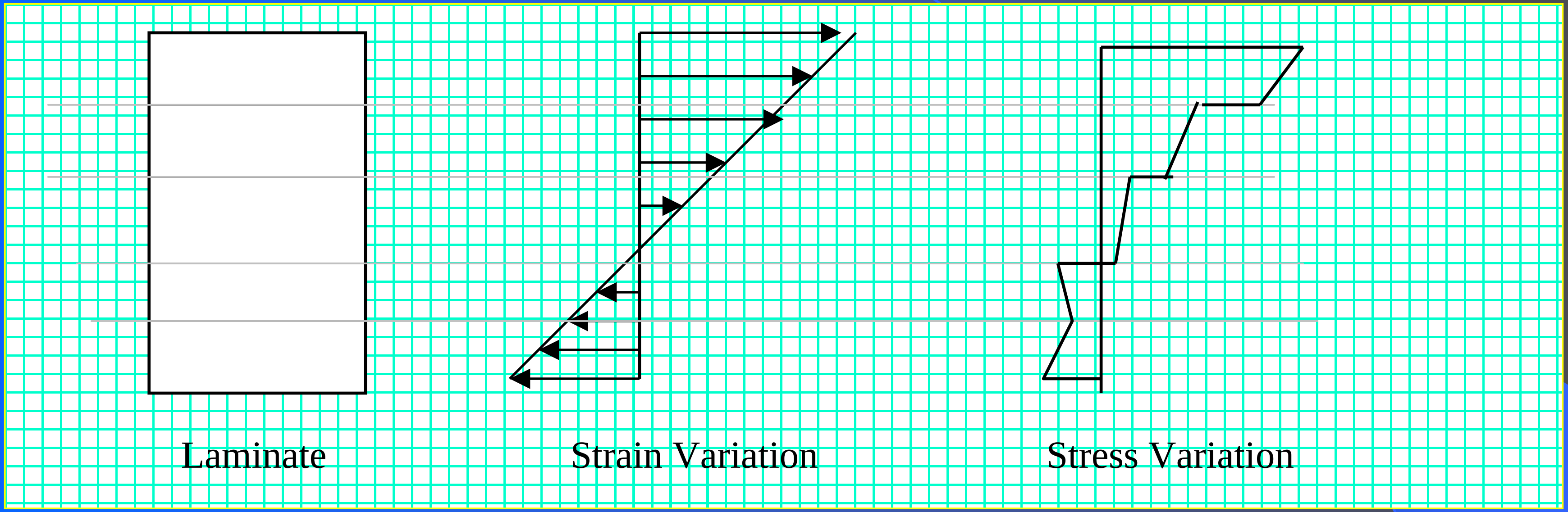
$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

The reduced transformed stiffness matrix, Q_{xy} corresponds to that of the ply located at the point along the thickness of the laminate.

Substituting the previous result,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

Strain and Stress in a Laminate



- The stresses vary linearly only through the thickness of each lamina.
- The stresses may jump from lamina to lamina since the transformed reduced stiffness matrix changes from ply to ply.

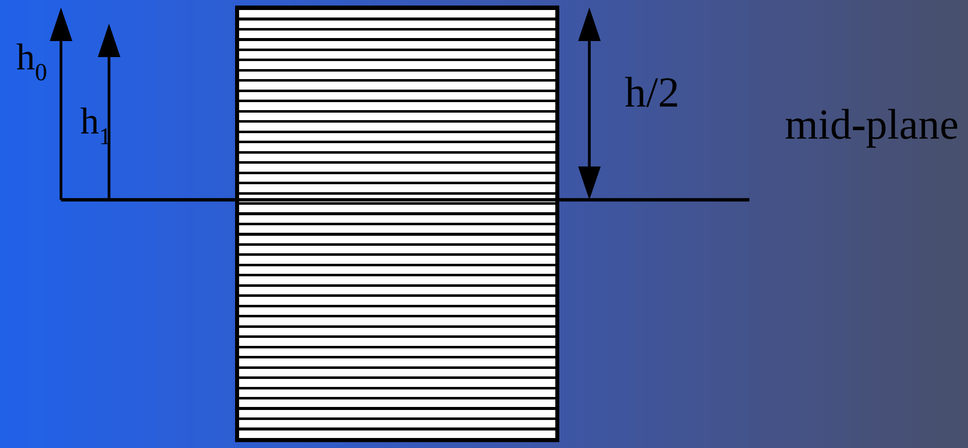
Strain and Stress in a Laminate

- These global stresses can then be transformed to local stresses through the Transformation equation.
- Likewise, the local strains can be transformed to global strains.
- Can then be used in the Failure criteria discussed previously.
- *All that remains is how to find the midplane strains and curvatures of a laminate if the applied loading is known?*

Force and Moment Resultants

- The stresses in each lamina can be integrated to give resultant forces and moments (or applied forces and moments.)
- Since the forces and moments applied to a laminate will be known, the midplane strains and plate curvatures can then be found.
- Consider a laminate made of *n* plies as shown, each ply has a thickness t_k.
- The location of the midplane is h/2 from the top or bottom surface.
- The z coordinate of each ply surface is given by

$$h_0 = \frac{h}{2} \text{ (top surface) and } h_1 = \frac{h}{2} - t_1 \text{ (bottom surface)}$$



Force and Moment Resultants

- Integrating the global stresses in each lamina gives the resultant forces per unit length in the *x*-*y* plane through the laminate thickness as

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz \qquad M_x = \int_{-h/2}^{h/2} \sigma_x z dz$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y dz \qquad M_y = \int_{-h/2}^{h/2} \sigma_y z dz$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz \qquad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz$$

N_x, N_y = normal force/unit length

N_{xy} = shear force/unit length

M_x, M_y = bending moment/unit length

M_{xy} = twisting moment/unit length

- Similarly, integrating the stresses in each lamina gives he resulting moments per unit length in the *x*-*y* plane through the thickness of the laminate.

Force and Moment Resultants

- In matrix form

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz \qquad \longrightarrow \qquad \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z dz \qquad \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z dz$$

- Substituting

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Force and Moment Resultants

- The resultant forces and moments can be written in terms of the midplane strains and curvatures

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} dz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} zdz$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} zdz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} z^2dz$$

Force and Moment Resultants

- Since the midplane strains and plate curvatures are independent of the z coordinate and the transformed reduced stiffness matrix is a constant for each ply —

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \int_{h_{k-1}}^{h_k} dz + \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \int_{h_{k-1}}^{h_k} z\,dz \right\}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \int_{h_{k-1}}^{h_k} z\,dz + \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \int_{h_{k-1}}^{h_k} z^2\,dz \right\}$$

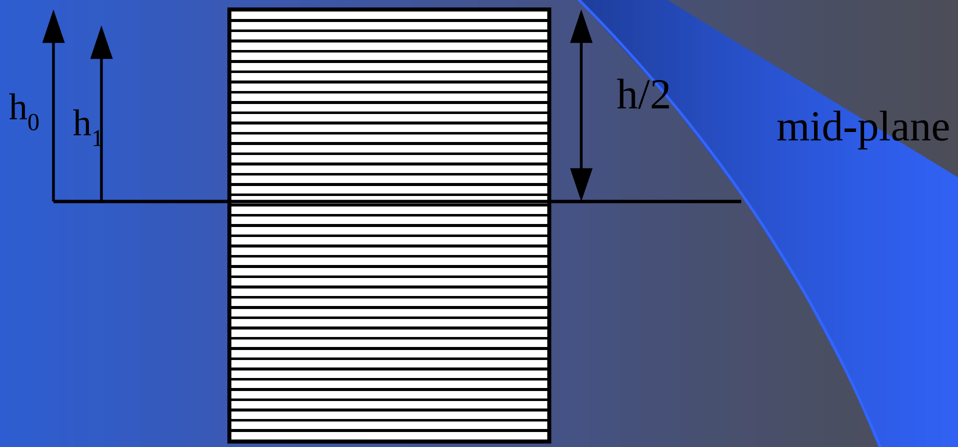
Force and Moment Resultants

- From the geometry (and a little calculus) we can solve the integrals

$$\int_{h_{k-1}}^{h_k} dz = \left(h_k - h_{k-1}\right)$$

$$\int_{h_{k-1}}^{h_k} z\,dz = \frac{1}{2}\left(h_k{}^2 - h_{k-1}{}^2\right)$$

$$\int_{h_{k-1}}^{h_k} z^2\,dz = \frac{1}{3}\left(h_k{}^3 - h_{k-1}{}^3\right)$$



Force and Moment Resultants

- Furthermore only the stiffnesses are unique for each layer, *k*. Thus, $[\varepsilon^0]_{x,y}$ and $[\kappa]_{x,y}$ can be factored outside the summation sign

$$[N]_{x,y} = \left[\sum_{k=1}^{n} [Q]_{x,y}^{k} (h_k - h_{k-1})\right][\varepsilon^0]_{x,y} + \left[\frac{1}{2}\sum_{k=1}^{n} [Q]_{x,y}^{k} \left(h_k^{\,2} - h_{k-1}^{\,2}\right)\right][\kappa]_{x,y}$$

$$[M]_{x,y} = \left[\frac{1}{2}\sum_{k=1}^{n} [Q]_{x,y}^{k} \left(h_k^{\,2} - h_{k-1}^{\,2}\right)\right][\varepsilon^0]_{x,y} + \left[\frac{1}{3}\sum_{k=1}^{n} [Q]_{x,y}^{k} \left(h_k^{\,3} - h_{k-1}^{\,3}\right)\right][\kappa]_{x,y}$$

- Define —

$$A_{ij} = \sum_{k=1}^{n} [Q]_{x,y}^{k} (h_k - h_{k-1}), \quad B_{ij} = \frac{1}{2}\sum_{k=1}^{n} [Q]_{x,y}^{k} \left(h_k^{\,2} - h_{k-1}^{\,2}\right), \quad D_{ij} = \frac{1}{3}\sum_{k=1}^{n} [Q]_{x,y}^{k} \left(h_k^{\,3} - h_{k-1}^{\,3}\right)$$

- [A], [B], [D] are called the extensional, coupling, and bending stiffness matrices, respectively.

Laminated Composite Analysis

$$[N]_{x,y} = [A_{ij}][\varepsilon^0]_{x,y} + [B_{ij}][\kappa]_{x,y}$$
$$[M]_{x,y} = [B_{ij}][\varepsilon^0]_{x,y} + [D_{ij}][\kappa]_{x,y}$$

Combine into one general expression for laminate composite analysis relating the in-plane forces and moments to the midplane strains and curvatures —

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

Laminated Composite Analysis

- The extensional stiffness matrix [*A*] relates the resultant in-plane force to the in-plane strains.
- The bending stiffness matrix [*D*] relates the resultant bending moments to the plate curvatures.
- The coupling stiffness matrix [*B*] relates the force and moment terms to the midplane strains and midplane curvatures.

Laminate Special Cases

- Symmetric: $[B] = 0$
  - Load-deformation equation and moment-curvature relation decoupled.
- Balanced: $A_{16} = A_{26} = 0$.
- Symmetric and Balanced:
  - Orthotropic with respect to inplane behavior.

$$\begin{bmatrix} N_x \\ N_y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{11} & A_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_x{}^0 \\ \varepsilon_y{}^0 \end{bmatrix}$$

$$N_{xy} = A_{66} \gamma_{xy}{}^0$$

Laminate Special Cases

- Cross-Ply: $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0.$
  - Some decoupling of the six equations.

$$\begin{bmatrix} N_x \\ N_y \\ M_x \\ M_y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{12} & A_{22} & B_{12} & B_{22} \\ B_{11} & B_{12} & D_{11} & D_{12} \\ B_{12} & B_{22} & D_{12} & D_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_x^{\ 0} \\ \varepsilon_y^{\ 0} \\ \kappa_x^{\ 0} \\ \kappa_y^{\ 0} \end{bmatrix}$$

$$\begin{bmatrix} N_{xy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{66} & B_{66} \\ B_{66} & D_{66} \end{bmatrix} \begin{bmatrix} \gamma_{xy}^{\ 0} \\ \kappa_{xy}^{\ 0} \end{bmatrix}$$

  - Orthotropic with respect to both inplane and bending behavior.

Laminate Special Cases

- Symmetric Cross-Ply:
  - [B] =0
  - $A_{16} = A_{26} = D_{16} = D_{26} = 0.$
  - Significant decoupling

$$\begin{bmatrix} N_x \\ N_y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{11} & A_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_x{}^0 \\ \varepsilon_y{}^0 \end{bmatrix} \qquad \begin{bmatrix} M_x \\ M_y \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{11} & D_{22} \end{bmatrix} \begin{bmatrix} \kappa_x{}^0 \\ \kappa_y{}^0 \end{bmatrix}$$

$$N_{xy} = A_{66} \gamma_{xy}{}^0 \qquad M_{xy} = D_{66} \kappa_{xy}{}^0$$

  - Orthotropic with respect to both inplane and bending behavior.

Laminated Composite Analysis

The following are steps for analyzing a laminated composite subjected to the applied forces and moments:

1. Find the values of the reduced stiffness matrix [Q_{ij}] for each ply.
2. Find the value of the transformed reduced stiffness matrix [Q_{xy}].
3. Find the coordinates of the top and bottom surfaces of each ply.
4. Find the 3 stiffness matrices [A], [B], and [D].
5. Calculate the midplane strains and curvatures using the 6 simultaneous equations (substitute the stiffness matrix values and the applied forces and moments).
6. Knowing the z location of each ply compute the global strains in each ply.
7. Use the stress-strain equation to find the global stresses.
8. Use the Transformation equation to find the local stresses and strains.

Laminate Compliances

- Since multidirectional laminates are characterized by stress discontinuities from ply to ply, it is preferable to work with strains which are continuous through the thickness.
- For this reason it is necessary to invert the load-deformation relations and express strains and curvatures as a function of applied loads and moments.

Laminate Compliances

- Performing matrix inversions

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} \\ a_{12} & a_{22} & a_{26} & b_{12} & b_{22} & b_{26} \\ a_{16} & a_{26} & a_{66} & b_{16} & b_{26} & b_{66} \\ c_{11} & c_{12} & c_{16} & d_{11} & d_{12} & d_{16} \\ c_{12} & c_{22} & c_{26} & d_{12} & d_{22} & d_{26} \\ c_{16} & c_{26} & c_{66} & d_{16} & d_{26} & d_{66} \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix}$$

or in brief

$$\begin{bmatrix} \varepsilon^0 \\ \hline \kappa \end{bmatrix} = \left[\begin{array}{c:c} a & b \\ \hdashline c & d \end{array}\right] \begin{bmatrix} N \\ \hline M \end{bmatrix}$$

Laminate Compliances

- Where [a], [b], [c], and [d] are the laminate extensional, coupling, and bending compliance matrices obtained as follows:

$$[a] = [A]^{-1} - \{[B^*][D^*]^{-1}\}[C^*]$$
$$[b] = [B^*][D^*]^{-1}$$
$$[c] = -[D^*]^{-1}[C^*] \text{ also } [c] = [b]^T$$
$$[d] = [D^*]^{-1}$$

and

$$[B^*] = -[A]^{-1}[B]$$
$$[C^*] = [B][A]^{-1}$$
$$[D^*] = [D] - \{[B][A]^{-1}\}[B]$$

- *NB: the compliances that relate midplane strains to applied moments are not identical to those that relate curvatures to in-plane loads.*

Engineering Constants for a Multi-Axial Laminate

- From the laminate compliances we can compute the engineering constants —

$$\overline{E_x} = \frac{1}{ha_{xx}} \qquad \overline{E_y} = \frac{1}{ha_{yy}} \qquad \overline{G_{xy}} = \frac{1}{ha_{ss}}$$

$$\overline{\nu_{xy}} = -\frac{a_{yx}}{a_{xx}} \qquad \overline{\nu_{yx}} = -\frac{a_{xy}}{a_{yy}} \qquad \overline{\eta_{sx}} = \frac{a_{xs}}{a_{ss}}$$

$$\overline{\eta_{xs}} = \frac{a_{sx}}{a_{xx}} \qquad \overline{\eta_{ys}} = \frac{a_{sy}}{a_{yy}} \qquad \overline{\eta_{sy}} = \frac{a_{ys}}{a_{ss}}$$

- As in UD lamina, symmetry implies —

$$\frac{\overline{\nu_{xy}}}{\overline{E_x}} = \frac{\overline{\nu_{yx}}}{\overline{E_y}}, \quad \frac{\overline{\eta_{xs}}}{\overline{E_x}} = \frac{\overline{\eta_{sx}}}{\overline{G_{xy}}}, \quad \frac{\overline{\eta_{ys}}}{\overline{E_y}} = \frac{\overline{\eta_{sy}}}{\overline{G_{xy}}}$$

Engineering Constants for a Multi-Axial Laminate

- Computational Procedure for Determination of Engineering Elastic Properties
  1. Determine the engineering constants of UD layer, E_1, E_2, ν_{12}, and G_{12}.
  2. Calculate the layer stiffnesses in the principal material axes, Q_{11}, Q_{22}, Q_{12}, and Q_{66}.
  3. Enter the fiber orientation of each layer, k.
  4. Calculate the transformed stiffnesses $[Q]_{x,y}$ of each layer, k.
  5. Enter the through thickness coordinates of the layer surfaces.
  6. Calculate the laminate stiffness matrices $[A]$, $[B]$, and $[D]$.
  7. Calculate the laminate compliance matrix $[a]$.
  8. Enter total laminate thickness, h.
  9. Calculate the laminate engineering properties in global, x, y axes.

Laminated Composite Analysis