Lesson 9

- Taylor-Proudman Theory
- Describe Taylor Columns
- Thermal Wind
- Recognize the thermal wind equation
- Basic understanding of derivation
- Be able to apply in ocean and atmosphere

Taylor-Proudman Theory

- In a homogenous fluid (ρ uniform), the geostrophic flow is 2D and does not vary in the direction of the rotation vector (Ω)
  - Does not allow any variation in wind as we move up the column - the distribution of wind near the surface is also the distribution in the upper atmosphere
    - i.e., zero wind shear \( \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \)
  - No information about vertical wind, only the of the variation in the horizontal wind in the vertical direction

Taylor-Proudman Theorem

- Assume:
  - Homogenous
  - Slow and steady flow (\( R_o << 1 \))
  - Friction is negligible
  - Horizontal EOM \( \rightarrow \) geostrophic balance
  - Vertical EOM \( \rightarrow \) hydrostatic balance
  - Thus a barotropic fluid (\( \nabla \times \text{EOM}_h \)), ρ=ρ(P)
- Result:
  \[
  (\Omega \cdot \nabla)\mathbf{v} = 0 \\
  \frac{\partial \mathbf{v}}{\partial z} = 0
  \]

Taylor Columns

- For slow, steady, frictionless flow of a barotropic fluid, the horizontal velocity cannot change in the direction of Ω, the flow is 2D
  - Vertical columns remain vertical (stiff)
  - Cannot be tilted or stretched
  - These are Taylor columns
Taylor Columns to Thermal Wind

- Since density does vary in the ocean and atmosphere, the Taylor-Proudman theorem must be modified
- The result is a diagnostic relation
  - The thermal wind equation

Thermal Wind: Concept

- Relates the vertical shear of the geostrophic wind to the horizontal temperature gradient
- Shows how temperature variations in the horizontal lead to vertical variations in the geostrophic wind velocity
- Assume, geostrophic and hydrostatic relations

Thermal Wind: Concept

- Westerly wind with a north-south temperature gradient
- Equation states we should have an increase of zonal winds with height (decrease of $T$ with latitude)
- Why do winds decrease with height above the troposphere?
Thermal Wind: Derivation

- Start with:
  - Geostrophic Equations
  \[ 0 = f_v - \frac{1}{\rho} \frac{\partial P}{\partial x} \quad 0 = -f_u - \frac{1}{\rho} \frac{\partial P}{\partial y} \]

- Then use:
  - Hydrostatic Equation
  \[ \frac{\partial P}{\partial z} = -\rho g \]

Thermal Wind: Derivation

- Result:
  \[ \frac{\partial}{\partial z}(\rho f v) = -g \frac{\partial \rho}{\partial x} \quad \frac{\partial}{\partial z}(\rho f u) = g \frac{\partial \rho}{\partial y} \]

- Note:
  - The pressure variations have been removed
  - The Coriolis parameter has no vertical dependence

Thermal Wind: Derivation

- Expand
  - The general form of Thermal Wind equations:
  \[ f_v \frac{\partial P}{\partial z} + f_p \frac{\partial \nu}{\partial z} = -g \frac{\partial \rho}{\partial x} \]
  \[ A \quad B \]
  \[ f_u \frac{\partial P}{\partial z} + f_p \frac{\partial \mu}{\partial z} = g \frac{\partial \rho}{\partial y} \]

- Compare (scale analysis) terms A and B

Thermal Wind (Current): Ocean

- Compare (scale analysis) terms A and B
  \[ f_v \frac{\partial P}{\partial z} + f_p \frac{\partial \nu}{\partial z} = -g \frac{\partial \rho}{\partial x} \]
  \[ A \quad B \]
  \[ f_u \frac{\partial P}{\partial z} + f_p \frac{\partial \mu}{\partial z} = g \frac{\partial \rho}{\partial y} \]
Thermal Wind (Current): Ocean

- Relationship for the ocean:
  \[
  \frac{\partial v}{\partial z} = -\frac{g}{f \rho} \frac{\partial \rho}{\partial x} \quad \frac{\partial u}{\partial z} = \frac{g}{f \rho} \frac{\partial \rho}{\partial y}
  \]
- Recall, \( u \) and \( v \) are geostrophic (\( u_g \) and \( v_g \))
- These relate the vertical shear of the geostrophic velocities to the horizontal gradients of density
- A general form for ocean

Thermal Wind (Current): Ocean

- Relationship in terms of \( T \) and \( S \)
  \[
  \frac{\partial v}{\partial z} = -\frac{g}{f} \left[ \alpha \frac{\partial T}{\partial x} - \beta \frac{\partial S}{\partial x} \right]
  \]
  \[
  \frac{\partial u}{\partial z} = -\frac{g}{f} \left[ \alpha \frac{\partial T}{\partial y} - \beta \frac{\partial S}{\partial y} \right]
  \]

Thermal Wind (Current): Ocean

- Example
  - \( S = 35 \text{ g/kg} \) is constant and \( T \) is observed to vary 2.5°C over 25 km in the east-west direction and \( v(z=0) = 1 \text{ m/s} \). What is \( v \) at \( z = -1000 \text{ m} \)?

- We can also calculate geostrophic velocities at one depth relative to another
  \[
  \frac{v_1 - v_2}{z_1 - z_2} = -\frac{g}{f \rho} \frac{\partial}{\partial x} (\rho_1 - \rho_2) \quad \frac{u_1 - u_2}{z_1 - z_2} = \frac{g}{f \rho} \frac{\partial}{\partial y} (\rho_1 - \rho_2)
  \]

Thermal Wind: Atmosphere

- Use hydrostatic equation and EOS to get:
  \[
  -\frac{RT}{P} = \frac{\partial P}{\partial z} = \frac{g \Delta z}{\partial P}
  \]
- Within two levels of constant pressure surfaces (thickness), the height difference is larger where the temperature is warmer
- Converse is also true
**Thermal Wind: Atmosphere**

- From our derivation:
  \[ f_{v} \frac{\partial \rho}{\partial z} + f_{p} \frac{\partial \nu}{\partial z} = -g \frac{\partial \rho}{\partial x} \]
  \[ A \quad B \]
  \[ f_{u} \frac{\partial \rho}{\partial z} + f_{p} \frac{\partial \mu}{\partial z} = g \frac{\partial \rho}{\partial y} \]

- *Cannot* neglect term A like in the ocean because the atmosphere is compressible

**Thermal Wind: Atmosphere**

- Thermal wind relations for the atmosphere:
  \[ \frac{\partial \mu}{\partial P} = -R \frac{\partial T}{\partial y} \]
  \[ \frac{\partial \nu}{\partial P} = R \frac{\partial T}{\partial x} \]

- Recall, \( u \) and \( v \) are geostrophic (\( u_{g} \) and \( v_{g} \))
- These relate the vertical shear of the geostrophic velocities to the horizontal gradients of temperature

**Thermal Wind Vector (\( \vec{v}_{t} \))**

- Thermal wind is parallel to the isotherms
  - Lower thickness to the left (right) in NH (SH)
  - I.e. warm air to the right facing downstream in NH
  - Now we can determine the sign of the geostrophic temperature advection by knowing the direction of the thermal wind

**Thermal Advection**

- Consider the geostrophic wind blowing at an angle to the isotherms so it advects temperature
  \[ -\vec{V}_{g} \cdot \nabla T \]
Thermal Advection: WAA
- $V_g$ turns clockwise (CW) with height
  - Veers
  - $V_W$ has a component from W to K
  \[-V_g \cdot \nabla T > 0\]

Thermal Advection: CAA
- $V_g$ turns counterclockwise (CCW) with height
  - Backs
  - $V_K$ has a component from K to W
  \[-V_g \cdot \nabla T < 0\]

Thermal Advection
- We can get an estimate of the horizontal temperature advection at a given location from a vertical wind profile from a sounding
- The geostrophic velocity at any level can be estimated from the mean temperature field provided we know the geostrophic velocity at another level

Thermal Advection
- Example
  - You are shipwrecked on an island in the central Atlantic (35N). One day you observe the clouds at three distinct levels in the atmosphere. The lowest level clouds are moving from north to south. The middle level clouds are moving from west to east and the upper level clouds are also moving from north to south. Assume all clouds are propelled by geostrophic winds. Describe the thermal wind and temperature advection.