Section 4.1

Mean of a Random Variable
Definition 4.1

Let $X$ be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of $X$ is

$$
\mu = E(X) = \sum_{x} x f(x)
$$

if $X$ is discrete, and

$$
\mu = E(X) = \int_{-\infty}^{\infty} x f(x) \, dx
$$

if $X$ is continuous.
**Theorem 4.1**

Let $X$ be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$
\mu_{g(X)} = E[g(X)] = \sum_{x} g(x) f(x)
$$

if $X$ is discrete, and

$$
\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx
$$

if $X$ is continuous.
**Definition 4.2**

Let $X$ and $Y$ be random variables with joint probability distribution $f(x, y)$. The mean, or expected value, of the random variable $g(X, Y)$ is

$$
\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y)
$$

if $X$ and $Y$ are discrete, and

$$
\mu_{g(x,y)} = E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) \, dx \, dy
$$

if $X$ and $Y$ are continuous.
Section 4.2

Variance and Covariance of Random Variables
Figure 4.1  Distributions with equal means and unequal dispersions
Definition 4.3

Let $X$ be a random variable with probability distribution $f(x)$ and mean $\mu$. The variance of $X$ is

$$\sigma^2 = E[(X - \mu)^2] = \sum_{x} (x - \mu)^2 f(x), \quad \text{if } X \text{ is discrete, and}$$

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx, \quad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, $\sigma$, is called the standard deviation of $X$. 
Theorem 4.2

The variance of a random variable $X$ is

$$\sigma^2 = E(X^2) - \mu^2.$$
Theorem 4.3

Let $X$ be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\sigma^2_{g(X)} = E\{[g(X) - \mu_g(x)]^2\} = \sum_x [g(x) - \mu_g(x)]^2 f(x)$$

if $X$ is discrete, and

$$\sigma^2_{g(X)} = E\{[g(X) - \mu_g(x)]^2\} = \int_{-\infty}^{\infty} [g(x) - \mu_g(x)]^2 f(x) \, dx$$

if $X$ is continuous.
Definition 4.4

Let $X$ and $Y$ be random variables with joint probability distribution $f(x, y)$. The covariance of $X$ and $Y$ is

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x \sum_y (x - \mu_X)(y - \mu_Y)f(x, y)$$

if $X$ and $Y$ are discrete, and

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y)f(x, y) \, dx \, dy$$

if $X$ and $Y$ are continuous.
Theorem 4.4

The covariance of two random variables $X$ and $Y$ with means $\mu_X$ and $\mu_Y$, respectively, is given by

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y.$$
Definition 4.5

Let $X$ and $Y$ be random variables with covariance $\sigma_{XY}$ and standard deviations $\sigma_X$ and $\sigma_Y$, respectively. The correlation coefficient of $X$ and $Y$ is

$$
\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.
$$
Section 4.3

Means and Variances of Linear Combinations of Random Variables
Theorem 4.5

If \( a \) and \( b \) are constants, then

\[
E(aX + b) = aE(X) + b.
\]
Corollary 4.1

Setting $a = 0$, we see that $E(b) = b$. 
Corollary 4.2

Setting $b = 0$, we see that $E(aX) = aE(X)$. 
Theorem 4.6

The expected value of the sum or difference of two or more functions of a random variable $X$ is the sum or difference of the expected values of the functions. That is,

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$
Theorem 4.7

The expected value of the sum or difference of two or more functions of the random variables $X$ and $Y$ is the sum or difference of the expected values of the functions. That is,

$$E[g(X, Y) \pm h(X, Y)] = E[g(X, Y)] \pm E[h(X, Y)].$$
Corollary 4.3

Setting \( g(X, Y) = g(X) \) and \( h(X, Y) = h(Y) \), we see that

\[
E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)].
\]
Corollary 4.4

Setting \( g(X, Y) = X \) and \( h(X, Y) = Y \), we see that

\[
E[X \pm Y] = E[X] \pm E[Y].
\]
Theorem 4.8

Let $X$ and $Y$ be two independent random variables. Then

$$E(XY) = E(X)E(Y).$$
Corollary 4.5

Let $X$ and $Y$ be two independent random variables. Then $\sigma_{XY} = 0$. 
Theorem 4.9

If $X$ and $Y$ are random variables with joint probability distribution $f(x,y)$ and $a$, $b$, and $c$ are constants, then

$$\sigma_{aX+bY+c}^2 = a^2 \sigma_X^2 + b^2 \sigma_Y^2 + 2ab \sigma_{XY}.$$
Corollary 4.6

Setting \( b = 0 \), we see that

\[
\sigma_{aX+c}^2 = a^2 \sigma_x^2 = a^2 \sigma^2.
\]
Corollary 4.7

Setting \( a = 1 \) and \( b = 0 \), we see that

\[
\sigma^2_{X+c} = \sigma^2_x = \sigma^2.
\]
Corollary 4.8

Setting $b = 0$ and $c = 0$, we see that

$$\sigma_{aX}^2 = a^2 \sigma_X^2 = a^2 \sigma^2.$$
Corollary 4.9

If $X$ and $Y$ are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$
Corollary 4.10

If $X$ and $Y$ are independent random variables, then

$$
\sigma_{aX+bY}^2 = a^2 \sigma_X^2 + b^2 \sigma_Y^2.
$$
Corollary 4.11

If $X_1, X_2, \ldots, X_n$ are independent random variables, then

$$\sigma_{a_1 X_1 + a_2 X_2 + \cdots + a_n X_n}^2 = a_1^2 \sigma_{X_1}^2 + a_2^2 \sigma_{X_2}^2 + \cdots + a_n^2 \sigma_{X_n}^2.$$
Section 4.4

Chebyshev’s Theorem
Figure 4.2 Variability of continuous observations about the mean
Figure 4.3 Variability of discrete observations about the mean
Theorem 4.10

(Chebyshev’s Theorem) The probability that any random variable $X$ will assume a value within $k$ standard deviations of the mean is at least $1 - 1/k^2$. That is,

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}.$$