Learning Objectives

- Analyze projectile motion - given the launch velocity in either magnitude-angle or unit-vector notation, calculate the particle's position, displacement, and velocity at a given instant during the flight and vice versa.

Kinematic Equations of Motion

The kinematic equations listed below can be used to analyze and solve two-dimensional problems involving the motion of objects falling under a constant gravitational acceleration.

- **Horizontal Motion**
  
  \[
  v_x = v_{0x} + a_x t \\
  x - x_0 = v_{0x} t + \frac{1}{2} a_x t^2 \\
  v_x^2 = v_{0x}^2 + 2a_x (x - x_0)
  \]

- **Vertical Motion**
  
  \[
  v_y = v_{0y} + a_y t \\
  y - y_0 = v_{0y} t + \frac{1}{2} a_y t^2 \\
  v_y^2 = v_{0y}^2 + 2a_y (y - y_0)
  \]

As noted earlier, the motion in \( x \) and \( y \) directions are independent of each other; that is, neither motion affects the other. A practical example of a 2-dimensional motion under gravity is projectile motion.

**Projectile motion** is the planar motion (pictured on the left) of an object that is launched with an initial velocity \( v_0 \). During its flight, the object's horizontal acceleration is zero and its vertical acceleration is the free-fall acceleration \(-g\) (Upward is taken to be a positive direction.) If \( v_0 \) is expressed as a magnitude (the speed \( v_0 \)) and an angle \( \theta_0 \) (measured from the horizontal), the particle's equations of motion along the horizontal \( x \) axis and vertical \( y \) axis are [noting that \( v_{0x} = v_0 \cos \theta_0 \) and \( v_{0y} = v_0 \sin \theta_0 \)]

- **Horizontal Motion:**
  
  Since \( a_x = 0 \), the \( x \) displacement is
  
  \[
  x - x_0 = v_{0x} t \\
  x - x_0 = (v_0 \cos \theta_0) t
  \]

- **Vertical Motion:**
  
  Since \( a_y = -g \), the \( y \) displacement is
  
  \[
  y - y_0 = v_{0y} t - \frac{1}{2} gt^2 \\
  y - y_0 = (v_0 \sin \theta_0) t - \frac{1}{2} gt^2
  \]

  And the vertical components of velocity;
  
  \[
  v_y = v_0 \sin \theta_0 - gt \\
  v_y^2 = (v_0 \sin \theta_0)^2 - 2g(y - y_0)
  \]

**Trajectory:** The trajectory (path) of a particle in projectile motion is parabolic. Combining the equations for the vertical and horizontal motion along with the simplifying assumptions that \( x_0 = 0 \) and \( y_0 = 0 \), we can show that the trajectory is;

\[
y = (\tan \theta_0)x - \frac{g}{2(v_0 \cos \theta_0)^2}x^2
\]

The object's horizontal range \( R \), which is the horizontal distance from the launch point to the point at which the object returns to the launch height, is

\[
R = \frac{v_0^2 \sin(2\theta_0)}{g}
\]

The horizontal range \( R \) is maximum for a launch angle of \( 45^\circ \). However, when the launch and landing heights differ, as in many sports, a launch angle of \( 45^\circ \) does not yield the maximum horizontal distance.
Problem 1:
A projectile is launched horizontally off a cliff that is 19 m high. It is launched with a speed of 11 m/s. (a) How long is the projectile in the air? (b) How far from the base of the cliff at ground level does it land? (c) What is its impact speed?

\[ V_{ox} = 11 \text{ m/s} \quad a_x = 0 \]
\[ V_{oy} = 0 \quad a_y = -g, \quad g = 9.8 \text{ m/s}^2 \]
\[ y_0 = 19 \text{ m} \]
\[ x_0 = 0, \quad y = 0 \]
\[ t = ? \]
\[ x = ? \]

(a) Using \( x - x_0 = V_{ox}t + \frac{1}{2}a_xt^2 \)
\[ -y_0 = -\frac{1}{2}gt^2 \]
\[ y_0 \quad t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2 \times 19}{9.8}} = 1.97 \text{ s} \]

The time between release and when it hit the ground is \( t = 1.97 \text{ s} \).

(b) Range (distance from base of cliff to point where it hit ground)
- Motion is x-direction

Again using, \( x - x_0 = V_{ox}t + \frac{1}{2}a_xt^2 \)
\[ \Rightarrow x = V_{ox}t = 11 \times 1.969 \]
\[ = 21.66 \text{ m} \]

(c) Since \( a_x = 0 \), \( V_{ox} = V_o \),

Using \( V_y = V_{oy} + a_yt = 0 - 9.8 \times 1.969 = -19.298 \text{ m/s} \)
\[ \Rightarrow \text{Total final velocity at impact} \quad V = \sqrt{V_x^2 + V_y^2} = \sqrt{11^2 + (-19.298)^2} \]
\[ = 22.21 \text{ m/s} \]
A soccer ball is kicked from the ground with an initial speed of 19.5 m/s at an upward angle of 45°. A player 55 m away in the direction of the kick starts running to meet the ball at that instant. What must be his average speed if he is to meet the ball just before it hits the ground?

For the player to meet the ball before it hits the ground, both the player and ball must arrive at the same position and take same amount of time to reach the point.

For the ball: (Vertical)
\[ y = y_0 = 0 \]
\[ v_{0y} = v_0 \sin \theta \]
\[ a_y = -g, \ g = 9.8 \text{ m/s}^2 \]
\[ t = ? \]
\[ \Rightarrow y - y_0 = v_{0y} t + \frac{1}{2} a_y t^2 \]
\[ 0 - 0 = 19.5 \sin 45° - \frac{1}{2} \times 9.8 \times t \]
\[ \Rightarrow t = \frac{(19.5 \sin 45°) \times 2}{9.8} \]
\[ = 2.814 \text{ s} \]

Therefore, the soccer player must travel a total distance \[ \Delta x_p = 55 - 38.8 \]
in total time of 2.814 s

\[ \Rightarrow \text{Average Speed} \]
\[ S_{avg} = \frac{\text{total distance}}{\text{total time}} \]
\[ = \frac{16.199}{2.814} \]
\[ = 5.76 \text{ m/s} \]

*The horizontal displacement of the ball*
\[ x - x_0 = v_{0x} t + \frac{1}{2} a_x t^2 \]
\[ x = 19.5 \cos 45° \times 2.814 = 38.8 \text{ m} \]
Problem 3:

Consider a rock thrown off a bridge of height 75 m at an angle $\theta = 25^\circ$ with respect to the horizontal as shown below. The initial speed of the rock is 15 m/s. Find the following quantities: (a) the maximum height reached by the rock, (b) the time it takes the rock to reach its maximum height, (c) the place where the rock lands, (d) the time at which the rock lands, and (e) the velocity of the rock (magnitude and direction) just before it lands.

First, calculate the time to reach max height noting that $V_y = 0$ at max height:

$\Rightarrow V_y = V_{oy} - gt$
$0 = V_0 \sin \theta - gt$
$\therefore t_{max} = \frac{V_0 \sin \theta}{g} = \frac{15 \sin 25^\circ}{9.8}$
$\Rightarrow t_{max} = 0.647 \text{ s}$

(a) $y_{max} = ?$, $y_0 = 75 \text{ m}$

Using $y - y_0 = V_{oy} t_{max} - \frac{1}{2} gt_{max}^2$

$\Rightarrow y_{max} - 75 = 15 \sin 25^\circ \times 0.647 - \frac{1}{2} \times 9.8 \times (0.647)^2$
$\Rightarrow y_{max} = 77.65 \text{ m}$

(b) $t_{max} = 0.647 \text{ s}$ (see above)

(c) First calculate the time to land; $y_{land} = 0$,

$y_{land} - y_0 = V_{oy} t_{land} + \frac{1}{2} a_{y} t_{land}^2$
\[ 0 - 75 = (15 \times \sin 25^\circ) t_{\text{land}} - 0.5 \times 9.8 \times t_{\text{land}}^2 \]

\[ 4.9 t_{\text{land}}^2 - 6.34 t_{\text{land}} - 75 = 0 \quad \text{(quadratic equation)} \]

\[ \Rightarrow t_{\text{land}} = 4.61 s \]

Thus, the rock lands (along x-axis) \( x_0 \)

\[ x - x_0 = (v_0 \cos \theta) t_{\text{land}} + \frac{1}{2} a t_{\text{land}}^2 \]

\[ \Rightarrow x = (15 \cos 25^\circ) \times 4.61 \]

\[ = 62.67 m \]

\( t_{\text{land}} = 4.61 s \) (See C above)

\[ v_{\text{land}} = 41 m/s \]

\( d \) Final velocity (with which it hits ground) has two components

* Since \( a_x = 0 \), \( v_x = v_{0x} = V_0 \cos \theta = 15 \cos 25^\circ = 13.60 m/s \)

* \( g = -9.8 m/s^2 \), \( v_y = v_{0y} - g t_{\text{land}} = V_0 \sin \theta - 9.8 \times 4.61 \)

\[ = 15 \sin 25 - 9.8 \times 4.61 = -38.80 m/s \]

\[ v = \sqrt{v_x^2 + v_y^2} = \sqrt{(13.6)^2 + (-38.8)^2} \]

\[ = 41 m/s \]

\( \theta = \tan^{-1} \left( \frac{v_y}{v_x} \right) = \tan^{-1} \left( \frac{-38.8}{13.6} \right) \]

\[ = -71^\circ \]
Problem 4:

A stone is projected at a cliff of height \( h \) with an initial speed of 42.0 m/s directed at angle \( \theta = 60.0^\circ \) above the horizontal. The stone strikes at \( A \), 5.50 s after launching. Find (a) the height \( h \) of the cliff, (b) the speed of the stone just before impact at \( A \), and (c) the maximum height \( H \) reached above the ground.

(a) \( y_0 = 0 \)  
\[ y = h \]

Using \[ y - y_0 = V_{oy}t + \frac{1}{2}a_y t^2 \]
\[ h - 0 = (42 \sin 60) \times 5.5 - \frac{1}{2} \times 9.8 \times (5.5)^2 \]
\[ h = 51.83 \text{ m} \]

(b) Calculate the two components of the final speed

* Since \( a_x = 0 \), \[ V_x = V_{0x} = V_0 \cos \theta = 42 \cos 60^\circ = 21 \text{ m/s} \]
* \( a_y = -g, g = 9.8 \text{ m/s}^2 \), \[ V_y = V_{0y} - gt = 42 \sin 60 - 9.8 \times 5.5 \]
\[ = -17.53 \text{ m/s} \]

\[ V = \sqrt{V_x^2 + V_y^2} = \sqrt{(21)^2 + (-17.53)^2} \]
\[ = 27.36 \text{ m/s} \]

(c) At max height, \( V_y = 0 \). We also know that \( y_0 = 0, y = H \)

Using \[ V_y^2 = V_{0y}^2 + 2ay (y - y_0) \]
\[ 0 = (V_0 \sin \theta)^2 - 2 \times 9.8 \times H \]
\[ 06 \quad H = \frac{(42 \sin 60)^2}{2 \times 9.8} = 67.5 \text{ m} \]