Department of Physics

United States Naval Academy

Lecture 31: Maxwell's Equations & Magnetism of Matter

Learning Objectives

- List all four Maxwell's equations and state the purpose of each. In particular, identify that in a capacitor that is being charged or discharged, a displacement current is said to be spread uniformly over the plate area, from one plate to the other.

In the last several lectures we studied various aspects of electric and magnetic fields. We learned that when the fields don't vary with time, such as an electric field produced by charges at rest or the magnetic field of a steady current, we can analyze the electric and magnetic fields independently without considering interactions between them. But when the fields vary with time, they are no longer independent. Faraday's law tells us that a time-varying magnetic field acts as a source of electric field, as shown by induced emfs in inductors. Ampere's law, including the displacement current discovered by James Clerk Maxwell, shows that a time-varying electric field acts as a source of magnetic field. This mutual interaction between the two fields is summarized as ***Maxwell's equations***. Maxwell's equations thus represent one of the most elegant and concise ways to state the fundamental laws of electricity and magnetism - they are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electromagnetism, classical optics, and electric circuits.

(a) **Gauss' Law for Electric Fields**: The electric flux out of any closed surface is proportional to the total charge enclosed within the surface.

$$\boxed{\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_o}} \tag{1}$$

(b) **Gauss' Law for Magnetic Fields**: The simplest magnetic structures are magnetic dipoles. Magnetic monopoles do not exist (as far as we know). Gauss' law for magnetic fields,

$$\boxed{\oint \vec{B}\cdot d\vec{A} = 0} \tag{2}$$

states that the net magnetic flux through any (closed) Gaussian surface is zero. It implies that magnetic monopoles do not exist.

(c) **Faraday's law**: In Lecture 26, we showed that a ***changing magnetic flux induces an electric field***, and rewrote Faraday's law of induction accordingly:

$$\boxed{\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}} \tag{3}$$

Similarly, it can be shown by symmetry arguments that a ***changing electric flux induces a magnetic field*** $\vec{B}$. This is **Maxwell's law**:

$$\oint \vec{B}\cdot d\vec{s} = \mu_o\varepsilon_o\frac{d\Phi_E}{dt}$$

This law relates the magnetic field induced along a closed loop to the changing electric flux Φ_E through the loop.

(d) **Maxwell-Ampere Law**: Ampere's law, $\oint \vec{B}\cdot d\vec{s} = \mu_o i_{enc}$, gives the magnetic field generated by a current i_{enc} encircled by a closed loop. Maxwell's law and Ampere's law can be combined into a single equation known as the Maxwell-Ampere Law;

$$\oint \vec{B}\cdot d\vec{s} = \mu_o\varepsilon_o\frac{d\Phi_E}{dt} + \mu_o i_{enc}$$

From this equation, we define the fictitious *displacement current* due to a changing electric field as

$$i_d = \varepsilon_o\frac{d\Phi_E}{dt}$$

and the Maxwell-Ampere law then becomes

$$\boxed{\oint \vec{B}\cdot d\vec{s} = \mu_o i_{d,enc} + \mu_o i_{enc}} \tag{4}$$

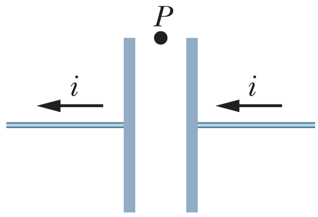
where $i_{d,enc}$ is the ***displacement current*** encircled by the integration loop. The idea of a displacement current allows us to retain the notion of continuity of current through a capacitor. While the displacement current is not a transfer of charge, we note that the real current i charging the capacitor and the displacement current i_d between the plates have the same value:

$$i_d = i$$

Equations (1) through (4) are the four fundamental equations of electromagnetism, called ***Maxwell's equations***.

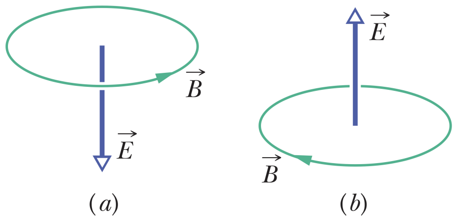
Question 1: CH32-Q2

The figure shows a parallel-plate capacitor and the current in the connecting wires that is discharging the capacitor. Are the directions of (a) electric field $\vec{E}$ and (b) displacement current i_d leftward or rightward between the plates? (c) Is the magnetic field at point P into or out of the page?



Question 2: CH32-Q3

The figure below shows, in two situations, an electric field vector $\vec{E}$ and an induced magnetic field line. In each, is the magnitude of $\vec{E}$ increasing or decreasing?

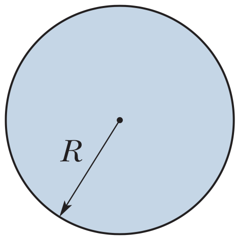


Problem 1: CH32-3

A Gaussian surface in the shape of a right circular cylinder with end caps has a radius of 12.0 cm and a length of 80.0 cm. Through one end there is an inward magnetic flux of 25.0 μWb. At the other end there is a uniform magnetic field of 1.60 mT, normal to the surface and directed outward. What are the (a) magnitude and (b) direction (inward or outward) of the net magnetic flux through the curved surface?

Problem 2: CH32-P7/10

The figure below shows a circular region of radius $R = 3.00$ cm in which a uniform electric flux is directed out of the plane of the page. The total electric flux through the region is given by $\Phi_E = (3.00\,\text{mV}\cdot\text{m/s})t$, where t is in seconds. What is the magnitude of the magnetic field that is induced at radial distances (a) 2.00 cm and (b) 5.00 cm? If the field is nonuniform and with magnitude given by $E = (0.500\,\text{V/m}\cdot\text{s})(1 - r/R)t$, where t is in seconds and r is the radial distance $(r \leq R)$. What is the magnitude of the induced magnetic field at radial distances (c) 2.00 cm and (d) 5.00 cm?



Problem 3: CH32-P17

A silver wire has resistivity $\rho = 1.62 \times 10^{-8}\ \Omega \cdot \mathrm{m}$ and a cross-sectional area of 5.00 mm^2. The current in the wire is uniform and changing at the rate of 2000 A/s when the current is 100 A. (a) What is the magnitude of the (uniform) electric field in the wire when the current in the wire is 100 A? (b) What is the displacement current in the wire at that time? (c) What is the ratio of the magnitude of the magnetic field due to the displacement current to that due to the current at a distance r from the wire?