Ch 5: Section 6

The Finite Well
Finite Well

* Inside $U(x) = 0$
* Outside $U(x) = U_o$
Inside the Finite Well

\[-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + U(x)\psi(x) = E\psi(x)\]

In the well \(U = 0\),

\[-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} = E\psi(x)\]

Re-write as we did for the infinite well:

\[\frac{d^2 \psi(x)}{dx^2} = -k^2 \psi(x) \text{ where } k^2 = \frac{2mE}{\hbar^2}\]
General solution to \( \frac{d^2 \psi(x)}{dx^2} = -k^2 \psi(x) \) is
\[
\psi(x) = A \sin(kx) + B \cos(kx)
\]
(For infinite well, we said there wave function outside must be zero. We cannot get the cosine part to be zero at \( x = 0 \). So we dismissed the cosine part in the case of an infinite well.)
Outside the Finite Well

Outside \( U(x) = U_0 \)

\[
\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + U(x)\psi(x) = E\psi(x)
\]

Rewrite as

\[
\frac{d^2 \psi(x)}{dx^2} = \frac{2m(U_0 - E)}{\hbar^2} \psi(x) \quad \text{and} \quad \alpha^2 = \frac{2m(U_0 - E)}{\hbar^2}
\]
Outside the Finite Well

\[
\frac{d^2\psi(x)}{dx^2} = \alpha^2 \psi(x)
\]

The MATHEMATICAL SOLUTION is

\[
\psi(x) = Ce^{\alpha x} + Ge^{-\alpha x}
\]

where \(C\) and \(G\) are constants (see p. 161)

Physically, the wave function cannot diverge as \(x \to \pm\infty\). So

\[
\psi(x) = Ce^{\alpha x} \text{ in II} \\
\psi(x) = Ge^{-\alpha x} \text{ in III}
\]
Applying Boundary Conditions

So far, we have the form of solution in three regions, and we have 4 constants. We find information about these constants as well as about the energy levels by applying the boundary conditions. The wave function and its first derivative must be smooth.
Energy Levels in Finite Well

\[ U_0 = \begin{cases} \frac{\hbar^2 k^2}{2m} \sec^2 \frac{kL}{2} & n = 1, 3, 5 \ldots \\ \frac{\hbar^2 k^2}{2m} \csc^2 \frac{kL}{2} & n = 2, 4, 6 \ldots \end{cases} \]

where \((n - 1)\pi < kL < n\pi\)

- \(U_0\) is the depth of the well (a constant)
- Only certain values of \(k\) (or \(E\)) are allowed (where \(U_0\) crosses).
- Depth of well determines how many energy states are allowed. (At least one.)
Compare Finite to Infinite well

**Finite**
- Energy is quantized.
- Ground state is NOT zero.
- Can have finite number of energy levels determined by depth \( U_0 \).
- Levels are lower (for same \( n \)).
- Wave function is exponential decaying outside

**Infinite**
- Energy is quantized.
- Ground state is NOT zero.
- Can have infinite number of energy levels.
- Levels are higher (for same \( n \)).
- Wave function is zero outside
The fact that the wave function is an exponential decay outside the finite well (as opposed to being zero) means the particle penetrates the walls of the well.

The penetration depth is

\[ \delta \equiv \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0 - E)}} \]
Summary of Finite Well Equations (memorize)

- $k^2 = \frac{2mE}{\hbar^2}$ (same as infinite well)
- $\alpha^2 = \frac{2m(U_0-E)}{\hbar^2}$
- $\delta \equiv \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0-E)}}$

- $\psi(x) = Ce^{\alpha x}$
- $\psi(x) = A \sin(kx) + B \cos(kx)$
- $\psi(x) = Ge^{-\alpha x}$

Diagram: Finite well potential $U = U_0$ for $0 < x < L$, $U = 0$ for $x = 0$ and $x = L$. Energy $E = KE$.
Standing waves with quantized $\lambda$ and $k$:

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L} \text{ and } k^2 = \frac{2mE}{\hbar^2}$$

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin k_n x \text{ (inside)}$$

$$\psi(x) = 0 \text{ (outside)}$$

And quantized energy

$$E_n = \frac{1}{2m} \left( \frac{n\pi\hbar}{L} \right)^2$$
Homework

* Ch 5: 34 due Thursday (05NOV15)

* Ch 5: 36, 37 and 42 in class