Detection Theory

The criterion for detection requires that the amount of sound energy collected by the receiver must exceed a threshold level to register a detection. The most common way to do this is first to express the ratio of signal to noise in decibels where:

\[ \text{SNR} = 10 \log \left( \frac{\text{Signal}}{\text{Noise}} \right) \]

The minimum SNR that is required to determine that there is a signal present in the environment a pre-established percentage of the time, is called the detection threshold, DT. The goal of this lesson is to be able to determine by calculation, a DT for our sonar system. It should be apparent that detection threshold is a statistical concept since the background noise that masks our signal fluctuates in randomly in time. Because of this, we will have to discuss some statistics ideas before we can calculate our Detection Threshold.

Threshold setting

Let's assume that in our environment there is random noise. Let us also assume we have a sonar system with a hydrophone that converts incident acoustic pressure into a voltage sent to the sonar processor. A plot of the voltage output from a hydrophone in an environment with noise might look something like that in figure 1.

![Figure 1 - 2 Volt Random Noise](image)

Now let's assume that there is a signal also present in the same environment as shown in figure 2 where the hypothetical signal is plotted without noise.
Figure 3 is a depiction of the sum of the signal plus noise. The question then becomes, what detection threshold should be set in the sonar system so that the signal can be detected through the background noise.

Should the threshold voltage be set at voltage $V_1$ where only the signal that is well above the noise will cause a detection? Or should it be $V_2$ where not only will some of the signals be detected but also some of the noise will cause a false detection? Or should it be $V_3$ where a good portion of the noise as well as most of the signals cause detections? What threshold voltage to set is a very difficult question to answer. The more important question though is, what percentage of the time can we tolerate a false alarm and tolerate missing a detection. Both circumstances are directly related to one another.
**Binary Decision Table**

Though there are only two possible answers at any moment for the conclusion that there is a signal present or not, there are two possible outcomes for both answers, the conclusion was either correct or incorrect. This is best summed up by the following two tables:

<table>
<thead>
<tr>
<th>Actual input</th>
<th>Decision/Hypothesis</th>
<th>Signal present</th>
<th>Signal not present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signal present</td>
<td>Correct detection</td>
<td>p(D)</td>
<td>Missed detection</td>
</tr>
<tr>
<td>Signal not present</td>
<td>False Alarm</td>
<td>p(FA)</td>
<td>Correct no detection</td>
</tr>
</tbody>
</table>

In this Matrix presentation, statisticians call the “decision” the “hypothesis.” It should be clear that for a given situation, the two hypothesis are mutually exclusive. If a signal is actually present, either hypothesis “signal present” or hypothesis “signal not present” must be selected. We are not allowing for an unknown hypothesis. Because of this, the sum of the probability that the “signal is present” and the probability that the “signal is not present” must add up to one.

There are two desired outcomes. We hope that anytime a signal actually exists, we chose the “signal present” hypothesis. Otherwise we have selected a “false negative” and have missed a valid target. On the other hand, if there is no signal present, we hope to always select the “signal not present” hypothesis. In this case, selecting “signal present” would be a “false positive” and would represent a false alarm. The below chart summarizes this idea.

<table>
<thead>
<tr>
<th>When there is noise only</th>
<th>When there is signal and noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decision</td>
<td></td>
</tr>
<tr>
<td>noise only</td>
<td>signal + noise</td>
</tr>
<tr>
<td>correct</td>
<td>wrong</td>
</tr>
<tr>
<td>p(null)</td>
<td>p(FA)</td>
</tr>
<tr>
<td>comments</td>
<td>you are correct, continue</td>
</tr>
<tr>
<td></td>
<td>searching</td>
</tr>
</tbody>
</table>

15-3
Probability Density Function

A better representation of the voltage output of a hydrophone to be used in determining the threshold setting is to plot the probability density function of the voltage. The probability density function represents the number of times the voltage was at a certain voltage (represented on the x-axis) per unit time. For good empirical reasons, we mathematically model the noise as a normal or “Gaussian” distribution of voltages about a mean value, $\mu$. The mathematical description of a Gaussian probability distribution function (PDF) is:

$$p(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-\mu)^2}{2\sigma^2}}$$

This equation tells us the probability of a particular value of voltage occurring in an interval of time. For each distribution we define the variance, $\sigma^2$. Variance tells us how much the distribution of the voltage “varies” about the mean value.

$$\sigma^2 = \frac{\int (v-\mu)^2 \, dv}{\int dv}$$

Standard deviation, $\sigma$, is the square root of the variance. We say that the probability that the voltage will lie within one standard deviation of the mean is about 67%. More exactly,

$$0.67 = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\sigma}^{\sigma} e^{-\frac{(v-\mu)^2}{2\sigma^2}} \, dv$$

Figure 4 above represents the probability density function of the receiver voltage for gaussian background noise only. The x-axis represents the voltage output of the hydrophone and the y-axis represents the probability that the voltage was at the level on the x-axis. This is roughly the same as the percentage of samples the sonar system will get at a particulate value of voltage in a particular time interval. From the curve, depending on where the threshold level, $v_o$,
is set, the shaded area under the curve and to the right of the threshold represents the probability of getting a false alarm. Since the total area under the curve represents 100% of the time, the remaining area represents, \( p(\text{null}) \), the probability that there is no signal.

For simplicity, we will often shift the distribution of noise such that it has a mean value of zero. With this shift, we could calculate \( P(\text{FA}) \),

\[
p(\text{FA}) = \frac{1}{\sigma \sqrt{2\pi}} \int_{v_0}^{\infty} e^{-\frac{(v-v_0)^2}{2\sigma^2}} \, dv
\]

Figure 5  -Signal + Noise

If we look at the probability density function of a signal plus gaussian background noise, we get a distribution like Figure 5. This curve is shifted to the for exactly the same reason that the curve in figure 3 is shifted up while the signal is present. From this curve, the shaded area under the curve represents the probability that a detection \( p(D) \) will occur. This probability can be calculated as follows:

\[
p(D) = \frac{1}{\sigma \sqrt{2\pi}} \int_{v_0}^{\infty} e^{-\frac{(v-H_{S+N})^2}{2\sigma^2}} \, dv
\]

The area to the left of the threshold voltage represents the probability of a missed detection, \( p(\text{Miss}) \).

**Detection Index**

Now to relate the time varying magnitude of the hydrophone voltage due to noise, to the time varying magnitude of the signal plus the noise, we define the quantity, \( d \), the detection index. The detection index can be thought of as the "processed signal to noise ratio". That is the
ratio of the signal to the noise after the sound energy has been converted to a voltage level and processed electronically. The formula for the detection index is:

\[ d = \frac{(\mu_{s+n} - \mu_n)^2}{\frac{1}{2}(\sigma^2_{s+n} + \sigma^2_n)} \]

where \( \mu \) is the mean voltage of the signal plus noise (s+n) or of the noise (n) (denoted by the subscript), and \( \sigma^2 \) is the variance. An example of the detection index for two PDF curves is shown below.
Compare the detection indices for the two PDF's shown above. Notice that the higher the ratio of signal to noise, the higher the detection index or in other words, the more likely a detection will occur at a particular threshold. Since the noise is the same in both cases, the probability of false alarm is the same for both detection indices. But since the d = 8 case is shifted to the right, more area is under the signal + noise curve to the right of the threshold. This tells us the probability of detection is greater in this case.

**Receiver Operating Characteristic (ROC) Curves**

To put all the preceding information together we can plot the probability for detection as a function of the probability of false alarm for various detection indices. The ROC curves are a set of curves that make our lives simpler by allowing us to be able to determine the probabilities for a sonar system for various signal to noise ratios. An example set of ROC curves for an ideal receiver system is shown below.

![ROC Curves](image)


This plot shows that for a given detection index, d, (which is the "processed signal to noise ratio"), that choosing a probability of detection determines the probability of false alarm or vice versa. Understand also that these ROC curves are dependent on the sonar system being
analyzed and will look different for a real sonar system compared to the idealized curves shown above.

One characteristic common to all ROC curves is the detection index line labeled $d = 0$, called the chance diagonal. With improved signal to noise ratio, the series of curves moves up and to the left of the chance diagonal corresponding to improved probability of detection, $p(D)$, and fewer false alarms, $p(FA)$.

**Detection Threshold**

The ROC curves discussed above are important in that from these curves, we can determine a good detection threshold, $DT$ in dB, for our sonar system. As a first step, you must decide (or be provided) the necessary detection probability you desire. This must be balanced with a reasonable probability of false alarms. It does you no good to insist on perfect probability of detection if your sonar system is constantly crying wolf with false alarms. Often the probability of detection specified is as low as 50%.

As an example, consider a required $p(D) = 50\%$ and a $p(FA) = 0.2\%$. The necessary detection index is then 9. Conversely, if the relationship between signal and noise is such that $d = 4$, then a probability of detection of 70% can not be obtained without accepting a value of 10% for $p(FA)$.

Imagine yourself in a noisy stadium at the concert of the year by your favorite artist. Can you hear what your friend is trying to tell you? Well that depends on many things including how loud the concert is as well as how loud your friend is talking. One other thing that can help you though is whether you see their lips moving or not. If you can "correlate" their lip movement to what little that you do hear from them, it is easier to tell what they are saying. The same holds true for sonar systems.

**Active Sonar System or Correlator Detector**

If we can compare the received signal and noise to a known signal, as in the example above, it will be easier to determine if there is an actual signal present or not. This is exactly what an active sonar system does. The active system sends out a signal with a known frequency, and pulse shape, and looks for a return signal with the same frequency and pulse shape through the background noise. Knowing this, we can better relate the detection threshold to the detection index. To find the equation for an ideal correlator detector, we must first review the meaning of detection index.

Previously we defined detection index,

$$d = \frac{(\mu_{s+n} - \mu_n)^2}{\frac{1}{2}(\sigma^2_{s+n} + \sigma^2_n)}$$

For the case of correlation, we might expect signal and noise to have the same variance. Detection index then is proportional to a ratio whose numerator is related to the average signal intensity and whose denominator is related to the average noise intensity.
Further we state without rigorous proof that the constant of proportionality is the number of samples, \( m \), obtained by the sonar system in a time period \( T \), called the integration time.

\[
d = m \frac{\langle S \rangle}{\langle N \rangle}
\]

A well known sampling theorem by Nyquist states that “the sampling rate must be at least twice the bandwidth, \( \omega \), of the received power so that no signal information is lost.” Nyquist’s theorem requires that the number of samples be at least, \( m = 2(\Delta f)T \). The average signal to noise ratio is then,

\[
\frac{\langle S \rangle}{\langle N \rangle} = \frac{d}{m} = \frac{d}{2(\Delta f)T}
\]

Detection threshold for a correlation detector is then defined the expected way band levels in dB are calculated.

\[
DT = 10 \log \frac{\langle S \rangle}{\langle N \rangle} = 10 \log \frac{d}{2(\Delta f)T}
\]

**Passive Sonar System or Energy Detector**

Imagine yourself at the same concert that we discussed above but now, your friend is facing away from you towards the stage. It would be much harder to determine what they were saying or even if they were talking to you, without the visual clue of seeing their lips moving. The same holds true for a passive sonar system. With a passive system, the operator is looking for a signal even though he does not know what type or frequency signal or even if there is one present. For this case, the equation for how the detection index relates to the signal and noise is different. For the passive sonar, we can show with some difficulty that \( d \) is given by the equation:

\[
d = (\Delta f)T \left( \frac{\langle S \rangle}{\langle N \rangle} \right)^2
\]

Again solving for the average signal to noise ratio,

\[
\frac{\langle S \rangle}{\langle N \rangle} = \left( \frac{d}{(\Delta f)T} \right)^{\frac{1}{2}}
\]

Detection threshold for a passive detector is then defined the expected way band levels in dB are calculated

\[
DT = 10 \log \frac{\langle S \rangle}{\langle N \rangle} = 10 \log \left( \frac{d}{(\Delta f)T} \right)^{\frac{1}{2}} = 5 \log \left( \frac{d}{(\Delta f)T} \right)
\]

** Note that this only holds true for small signal to noise ratios (\( S/N << 1 \)) and large sample sizes (\( (\Delta f)T >> 1 \)).
The Sonar System Detection Threshold

Now let's put this together starting with a very basic illustration of the components of a sonar system. This system is composed of an array of hydrophones, a receiver, a display and an operator. Each one of these components including the observer or operator contributes to the detection threshold of the system.


We have only discussed an idealized prediction of the detection threshold of the above system. Many other things will reduce the detectability of the system but we can not increase the detectability above the idealized case. Some of the items that can affect the systems detection threshold.

- Fluctuating signal from the target will degrade system performance. \( p(D) \) will be a function of amplitude density probability of signal. If signal follows a Rayleigh distribution it can be shown that \( p(D) \) can be approximated with threshold \( Y_0 \) and detection index \( d \).

\[
p(D) = \frac{e^{-\left(\frac{Y_0^2}{2 + d}\right)}}{1 + \frac{2}{d}}
\]

- valid if \( p_a > 0.1 \) and \( p_{fa} < 0.01 \)

- If there are more than one signal present.
- If there is multipath propagation.
- If bandwidth- time product (\( \omega T \)) is not greater than 1.
If post-detector averager or smoothing filter is used to remove noise from processor output.
We will leave the study of these factors to a more in depth study of sonar systems.
1. The curves below represent the probability density functions for the “voltage” distribution of signal plus noise and noise alone. The detection threshold is set at 60 mV. If the area under the curve shaded /// = 0.10 and the area under the curve shaded \/// = 0.30, calculate:
   a) \( p(D) \)
   b) \( p(\text{miss}) \)
   c) \( p(\text{FA}) \)
   d) \( p(\text{null}) \)
   e) The detection index

![Diagram of probability density functions](image)

2. A series of 5 processed voltage readings, for the case of noise alone, is tabulated below:
   Trial #   Processed “noise” voltage
   1         2
   2         1
   3         3
   4         2
   5         1

A series of processed voltage readings for the case of signal plus noise is tabulated below
   Trial #   Processed “noise” voltage
   1         2
   2         3
   3         4
   4         2
   5         3
a) Complete the following table and draw the receiver operating curve corresponding to the coordinate (p(FA), P(D)).

<table>
<thead>
<tr>
<th>Threshold Voltage</th>
<th>P(FA)</th>
<th>P(D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>3/5 = 0.6</td>
<td>5/5 = 1.0</td>
</tr>
<tr>
<td>2.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Compute the mean, $\mu_{s+n}$ and the standard deviation, $\sigma_{s+n}$ for the processed “signal + noise” case.

Note that sample variance is found from $\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n-1}$

c) Compute the mean, $\mu_n$ and the standard deviation, $\sigma_n$ for the processed “noise alone” case.

d) Compute the processed signal to noise ratio parameter, d
3. The human ear can be modeled as an energy detector (passive system) of bandwidth 50 Hz and integration time of 0.50 seconds. What will be the detection threshold for the ear given a 60% probability of detection and a 5% probability of false alarm?

4. A scuba diver must be able to hear a 1000 Hz tone in a background of ocean noise that is isotropic with a constant intensity spectrum level of 80 dB. If the ear is modeled as an energy detector with a 50 Hz bandwidth and an integration time of 0.5 sec, what is the minimum rms pressure in μPa necessary for him to hear the tone with a probability of detection of 50% and a probability of false alarm of 0.05%? Transmission loss is neglected here. Let the directivity index equal 3 dB.

5. A cross-correlator receiving (active) system is used to detect a known signal in a background of Gaussian noise. The predetermined criterion for detection is such that p(D) = 50% and p(FA) = 0.2%. Calculate the system’s detection threshold given that the signal duration is 200 milliseconds. The bandwidth is 100 Hz.

6. A surface ship is trying to prosecute an enemy submarine. If the surface ship’s sonar system has P(D) = 75% and P(FA) = 0.1%, what is the probability that a torpedo will be wasted on a false target?

7. A passive continuous line array sonar 30 m long receives signals in a one half octave bandwidth centered on a frequency of 400 Hz. The sonar’s receiver may be modeled as a passive energy detector with an integration time of 2.0 seconds. The line array is towed in an environment where the ambient noise spectrum level due to distant shipping is 51 dB, the ambient noise spectrum level due to wind driven waves is 54 dB, the self noise spectrum level is 52 dB, and the local sound speed is 1500 m/s. All spectrum levels are constant in the range of the frequencies in the sonar’s receiver bandwidth. What is the sonar’s figure of merit (FOM) against a target radiating white noise (with a spectrum level of 120 dB at the sonar’s center frequency) given a requirement for p(D) = 50.0% and p(FA) = 0.10%. The directivity index is given by DI = 10 log(2L/λ) where L is the array length and λ is the wavelength. Considering only spherical spreading and no attenuation (TL = 20 log r), solve for the detection range.
Lesson 15

Signal and Noise

Combined Signal and Noise

Figure 1 - 2 Volt Random Noise

Figure 2 - 2 Volt Signal with no Noise

Figure 3 - 2 Volt Random Noise with 2 Volt Signal

Combined Signal and Noise

SNR = 1

Binary Decision Table

<table>
<thead>
<tr>
<th>Decision/Hypothesis</th>
<th>Signal present</th>
<th>Signal not present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signal present</td>
<td>Correct detection</td>
<td>Missed detection</td>
</tr>
<tr>
<td>Signal not present</td>
<td>Alarm</td>
<td>No detection</td>
</tr>
</tbody>
</table>

Actual input

| Signal present | Correct detection | Missed detection |
| Signal not present | Alarm | No detection |

Normal or Gaussian Distribution

PDF

Normal or Gaussian Distribution

PDF – Signal and Noise

PDF – Signal and Noise

Probability Density Function - Noise

Normal or Gaussian Distribution

PDF

Normal or Gaussian Distribution
Lesson 15

Detectivity (Detection) Index

\[ d = \frac{(\mu_{signal} - \mu_{noise})^2}{\sqrt{2}(\sigma_{signal}^2 + \sigma_{noise}^2)} \]

“Processed SNR”

Example ROC Curve

\[ d = \frac{(\mu_{signal} - \mu_{noise})^2}{\sqrt{2}(\sigma_{signal}^2 + \sigma_{noise}^2)} \]

\[ \sigma_{signal}^2 = \sigma_{noise}^2 \]

Receiver Operating Curves

ROC Curves

Nyquist Theorem

- The sampling rate must be at least twice the bandwidth, \( \Delta f \), of the received power so that no signal information is lost.

Correlation Process

Noise or signal plus noise

\[ s_t = s + n \]

or \( s_t = n \)

Replica of signal

Threshold, \( V_a \)
Correlation Detector (active)

\[ d = \frac{\mu_s - \mu_n}{\sigma_s} = \frac{\langle S \rangle}{\langle N \rangle} \]

\[ d = m \frac{\langle S \rangle}{\langle N \rangle} \]

\[ \frac{\langle S \rangle}{\langle N \rangle} = \frac{d}{m} = \frac{d}{2(\Delta f)T} \]

\[ DT = 10\log \left( \frac{\langle S \rangle}{\langle N \rangle} \right) = 10\log \left( \frac{d}{2(\Delta f)T} \right) \]

Square Law (Energy Detector) Process

Energy Detector (passive)

\[ d = (\Delta f)T \left( \frac{\langle S \rangle}{\langle N \rangle} \right)^{1/2} \]

\[ \frac{\langle S \rangle}{\langle N \rangle} = \left( \frac{d}{(\Delta f)T} \right)^{1/2} \]

\[ DT = 10\log \left( \frac{\langle S \rangle}{\langle N \rangle} \right) = 10\log \left( \frac{d}{(\Delta f)T} \right)^{1/2} = 5\log \left( \frac{d}{(\Delta f)T} \right) \]