1. (25) A stretchy string of length, $L$, is suspended between two fixed posts. A template is created that stretches the string into the initial position shown below:

![Diagram of a string suspended between two posts with initial positions marked]

$$y(x, t = 0) = \begin{cases} 
+h & 0 < x < \frac{L}{2} \\
-h & \frac{L}{2} < x < L 
\end{cases}$$

A reasonable approximation for this initial condition is:

If the template is rapidly removed from this initial rest position, the string vibrates according to the general result:

$$y(x, t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin k_n x$$

a. (15) Find the expression for the coefficients $A_n$ and $B_n$.

b. (5) If $h = 1$ cm, find the amplitude of the fundamental mode and the first five overtones ($n = 1$ to $n = 6$). Sketch the non-zero modes.

c. (5) If the mass of the string is 0.030 kg and the length is 0.314 m and the tension is 5.0 N, how much energy is in the $n=2$ mode?

$$E_n = \frac{1}{4} m \omega_n^2 A_n^2$$

$$k_2 = \frac{2\pi}{0.314} = \frac{2\pi}{0.314} = 22.9 \%$$
2. (25) A long, Aluminum bar is fixed at one end \((x=0)\) and loaded with a large mass, \(M\), at the other end \((x=L)\). The bar has cross sectional area, \(S\).

The bar vibrates in the longitudinal mode subject to the wave equation,

\[
\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2}
\]

where \(c^2 = \frac{Y}{\rho}\). We showed that the complex harmonic solution of the wave equation is,

\[
\tilde{\xi}(x, t) = \tilde{A}e^{j(\omega t - kx)} + \tilde{B}e^{j(\omega t + kx)}.
\]

a. (10) What are the boundary conditions for this system.

b. (5) Use your boundary condition at \(x=0\) to show that: \(\tilde{\xi}(x, t) = -j2\tilde{A}e^{j\omega t} \sin kx\)

\[
\tilde{\xi}(x, t) = \tilde{A}e^{j\omega t} + \tilde{B}e^{j\omega t} = 0 \quad \Rightarrow \quad \tilde{A} = -\tilde{B}
\]

\(\tilde{\xi}(x, t) = \tilde{A}e^{j\omega t}(e^{-j kx} - e^{+j kx}) = -2j\tilde{A}e^{j\omega t} \sin kx\)

Q.E.D

c. (5) Combine with your boundary condition at \(x=L\) to show that: \(\cot kL = \frac{M \omega^2}{SYk}\)

d. (5) Since \(Y = \rho c^2 = \frac{m_{bar}}{SL} c^2\), show this can be rewritten: \(\cot kL = \frac{M}{m_{bar}} kL\)

\[-SY \frac{\partial T}{\partial x} \bigg|_{x=L} = M \frac{\partial^2 T}{\partial x^2} \bigg|_{x=L}\]

\[kS \left( \frac{\partial}{\partial x} \left( \frac{1}{2} \rho \left( \frac{\partial \tilde{u}}{\partial t} \right)^2 \sin kx \right) \right) = M \left( \frac{\partial}{\partial x} \left( \frac{\omega^2}{2 \rho} \sin kx \right) \right) = M \frac{\omega^2}{2 \rho} \sin kx \frac{\partial}{\partial x} \sin kx \]

\[kS \frac{\partial u}{\partial x} \sin kL = M \frac{\omega^2}{SYk} \sin kL\]

\[\cot kL = \frac{M \omega^2}{SYk}\]

Q.E.D.

\[
\cot kL = \frac{M \omega^2}{\left( \frac{m_{mass}}{S} \right) c^2 k} = \frac{M}{m_{bar}} \frac{\omega^2}{c^2} \frac{kL}{L} = \frac{M}{m_{mass}} kL
\]

Q.E.D.
3. (20) An Aluminum bar is 75.0 cm long with a circular cross section and a diameter of 1.00 cm. The bar is excited to vibrate in the bending mode where both ends are free.

We showed that Helmholtz equation for this bending mode is

\[ \frac{\partial^4 \Psi}{\partial x^4} = g^4 \Psi \]

with wave number, \( g = \frac{\omega}{v} \) and phase speed, \( v^2 = \omega \kappa \text{long} \). Further we showed that applying free-free boundary conditions to the solution to Helmholtz equation resulted in the following relation

\[ \cosh gL \cos gL = 1 \]

where

\[ gL = \frac{m \pi}{2} \quad \text{with} \quad m = 3.011, 4.9994, 7.00, 9.00, 11.0... \]

a. (5) What are the numeric values of the wave numbers for the first 3 normal modes? **Include units.**

b. (5) What are the first 3 normal mode wavelengths?

c. (5) What are the first 3 normal mode frequencies (in Hz)?

\[ 3.122 \]

d. (5) What are the phase speeds for the first 3 normal modes?
\[ g_L = \frac{m \pi}{2} \]

\[ g_1 = \frac{3.01 \pi}{2 (7.75 m)} = 2.007 \pi \text{ rad/m} = 6.31 \text{ rad/m} \]

\[ g_2 = \frac{5 \pi}{2 (7.75 m)} = 3.33 \pi \text{ rad/m} = 10.5 \text{ rad/m} \]

\[ g_3 = \frac{7 \pi}{2 (7.75 m)} = 4.67 \pi \text{ rad/m} = 14.7 \text{ rad/m} \]

\[ \chi_1 = \frac{2 \pi}{g_1} = \frac{2 \pi}{2.007 \pi \text{ rad/m}} = 0.996 \text{ m} \]

\[ \chi_2 = \frac{2 \pi}{g_2} = \frac{2 \pi}{3.33 \pi \text{ rad/m}} = 0.6 \text{ m} \]

\[ \chi_3 = \frac{2 \pi}{g_3} = \frac{2 \pi}{4.67 \pi \text{ rad/m}} = 0.429 \text{ m} \]

\[ g = \frac{m \pi}{2L} \quad \omega = g \sqrt{\frac{w K c_0}{g}} = g \sqrt{w K c_0} \]

\[ \chi = \frac{d}{4} = \frac{0.01 m}{4} = 0.0025 m \]

\[ c_e = 5050 \text{ m/s} \]

\[ \frac{\omega}{\sqrt{w}} = \sqrt{w} = g \sqrt{K c_0} \]

\[ \omega = g^2 K c_0 = \frac{m \pi^2}{4 L^2} K c_0 \]

\[ f = \frac{m^2 \pi K}{2 \pi (4L^2)} K c_0 \]

\[ = \frac{m^2 \pi K c_0}{8L^2} \]
\[ f_1 = \frac{(3.11)^2 \pi (0.0025 m)(5050\%)}{8 (0.75 m)^2} = 79.9 \text{ Hz} \]

\[ f_2 = \frac{(1.99^2 \pi)^2 (0.0025 m)(5050\%)}{8 (0.75 m)^2} = 220 \text{ Hz} \]

\[ f_3 = \frac{(7)^2 \pi (0.0025 m)(5050\%)}{8 (0.75 m)^2} = 432 \text{ Hz} \]

\[ U_1 = f_1 \lambda_1 = (79.9 \text{ Hz})(0.996 m) = 79.6 \text{ m/s} \]

\[ U_2 = f_2 \lambda_2 = (220 \text{ Hz})(0.6 m) = 132 \text{ m/s} \]

\[ U_3 = f_3 \lambda_3 = (432 \text{ Hz})(0.429 m) = 185 \text{ m/s} \]

or

\[ U_1 = \sqrt{2 \pi f_1 \lambda_0} = \sqrt{2 \pi (79.9 \text{ Hz})(0.0025 m)(5050\%)} = 79.6 \text{ m/s} \]

\[ U_2 = \sqrt{2 \pi f_2 \lambda_0} = 132 \text{ m/s} \]

\[ U_3 = \sqrt{2 \pi f_3 \lambda_0} = 185 \text{ m/s} \]
3. (25) An elastic membrane is stretched and clamped on a rigid circular frame with uniform tension. The fundamental frequency shown below is 70 Hz. The radius of the membrane is 0.10 m and the area density is 0.20 kg/m².

\[ f_{01} = \frac{\frac{1}{2} \pi a}{\pi a} = \frac{1}{2} \pi f_{01} \]

\[ c = \frac{2 \pi f_{01} a}{f_{01}} = \frac{2 \pi (70 \text{Hz})(1 \text{m})}{2 \pi} = 18.3 \text{ m/s} \]

a. (5) What is the phase speed of the standing wave on the membrane?

b. (5) What is the tension per unit length (assumed to be isotropic) of the elastic membrane?

\[ T = \sigma \frac{c}{c} = \frac{12.5 \pi}{2.4} \approx 47.2 \text{ N/m} \]

c. (10) What are the next 5 lowest frequencies at which this membrane can vibrate?

d. (5) Include a rough sketch of the oscillation spatial pattern similar to figure 4.4.1 in your textbook for each frequency.

\[ f_{11} = \frac{\pi c}{2 \pi a} = \left(70 \text{ Hz}\right)\left(\frac{3.83}{2.4}\right) = 112 \text{ Hz} \]

\[ f_{21} = 70 \text{ Hz} \left(\frac{5.14}{2.4}\right) = 150 \text{ Hz} \]

\[ f_{02} = 70 \text{ Hz} \left(\frac{5.52}{2.4}\right) = 161 \text{ Hz} \]

\[ f_{31} = 70 \text{ Hz} \left(\frac{6.38}{2.4}\right) = 186 \text{ Hz} \]

\[ f_{12} = 70 \text{ Hz} \left(\frac{7.02}{2.4}\right) = 205 \text{ Hz} \]