Maxwell's Equations

SP212 - General Physics II

Maxwell's equations are a set of four equations that fully describe how electric and magnetic fields are generated. They were formulated by James Clerk Maxwell in 1861.

Gauss's Law: $\oint_{\mathrm{S}} \vec{E} \cdot \hat{n}\, dA = \frac{Q_{\text{inside}}}{\epsilon_0}$

Faraday's Law: $\oint_{\mathrm{C}} \vec{E} \cdot d\vec{l} = -\frac{d\phi_{\mathrm{m}}}{dt}$

Gauss' Law for Magnetism: $\oint_{\mathrm{S}} \vec{B} \cdot \hat{n}\, dA = 0$

Ampere-Maxwell Law: $\oint_{\mathrm{C}} \vec{B} \cdot d\vec{l} = \mu_0 I_{\mathrm{C}} + \mu_0 \epsilon_0 \frac{d\phi_{\mathrm{e}}}{dt}$

We are already quite familiar with two (or maybe two and a half) of these equations.

Gauss's Law: $\oint_S \vec{E} \cdot \hat{n}\, dA = \dfrac{Q_{\text{inside}}}{\epsilon_0}$

Gauss's Law tells us that electric charges create electric fields. These field lines start at positive charges and end at negative charges.

Faraday's Law: $\oint_C \vec{E} \cdot d\vec{l} = -\dfrac{d\phi_{\text{m}}}{dt}$

Faraday's Law tells us that time-varying magnetic fields also create electric fields. These fields are non-conservative: their fields lines form closed loops.

Gauss's Law for Magnetism

$$\oint_S \vec{B} \cdot \hat{n}\, dA = 0$$

Gauss's Law for magnetism

- The net magnetic flux through any closed surface will always be zero.
- This implies that *magnetic monopoles* (an isolated north or south pole) do not exist.
- If you cut a magnet in half you will then have two magnets, each with a N and S pole, not separated N and S poles.
- Magnetic field lines always form closed loops with no beginning or end. Therefore the net number of field lines passing through a closed surface will always be zero.

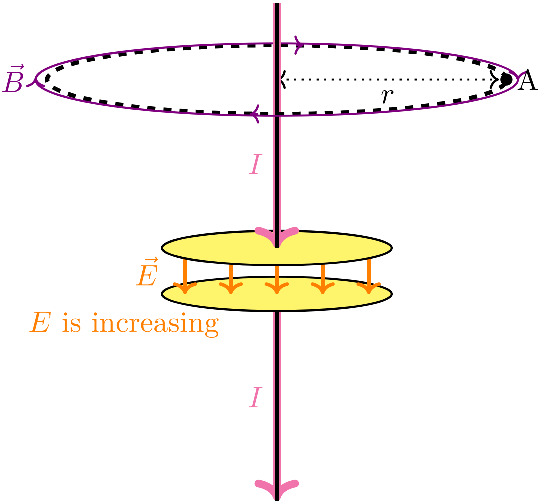
Our previous study of Ampere's law was valid for steady currents. It turns out that an additional term must be added for the equation to be universally valid.

$$\oint_{\mathrm{C}} \vec{B} \cdot d\vec{l} = \mu_0 I_{\mathrm{C}} + \mu_0 \epsilon_0 \frac{d\phi_{\mathrm{e}}}{dt}$$ Ampere-Maxwell Law

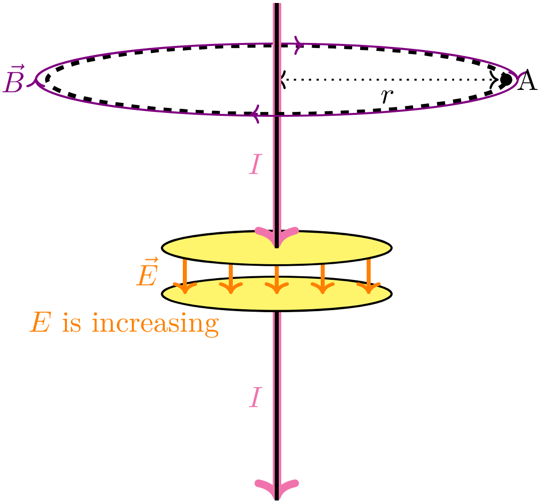
$$I_{\mathrm{d}} = \epsilon_0 \frac{d\phi_{\mathrm{e}}}{dt}$$ displacement current

This reduces to $\oint_{\mathrm{C}} \vec{B} \cdot d\vec{l} = \mu_0 (I_{\mathrm{C}} + I_{\mathrm{d}})$: the line integral of the magnetic field around a closed curve is equal to μ_0 multiplied by the sum of the current passing through the curve plus the displacement current passing through the curve, where displacement current is ϵ_0 multiplied by the time rate of change of the electric flux.

The Ampere-Maxwell law tells us that there are two ways to make magnetic fields: a current creates a magnetic field, and so does a time-varying electric field.



We have a parallel plate capacitor that is being charged by a current I moving downwards (towards the capacitor top plate). Position A is a distance r from the wire, far from the capacitor. What is the magnetic field at position A?



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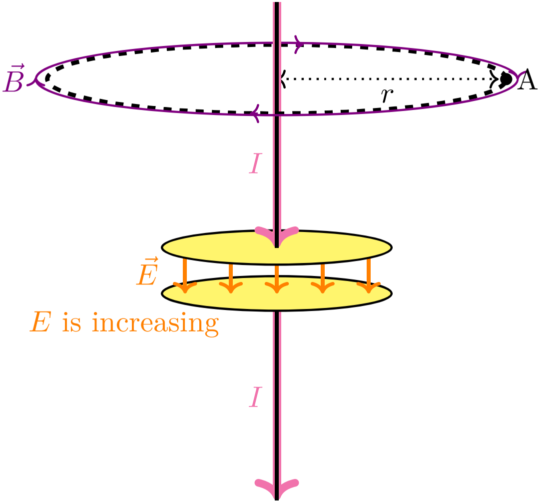
$$\oint_{\mathrm{C}} \vec{B} \cdot d\vec{l} = \mu_0 I_{\mathrm{C}} + \mu_0 \epsilon_0 \frac{d\phi_{\mathrm{e}}}{dt}$$

We draw an Amperian loop that is a circle of radius r center on the wire. We have rotational symmetry, so B is the same everywhere on the loop and the B-field lines make circles. Therefore $\oint_{\mathrm{C}} \vec{B} \cdot d\vec{l} = B2\pi r$.

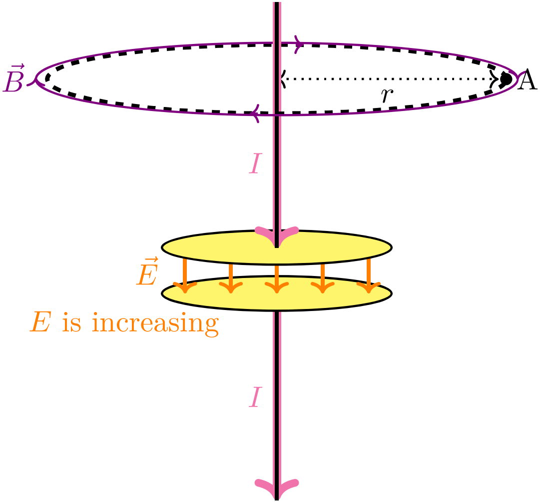


What is the magnetic field at position A?

$$\oint_{\mathrm{C}} \vec{B} \cdot d\vec{l} = B2\pi r = \mu_0 I_{\mathrm{C}} + \mu_0 \epsilon_0 \frac{d\phi_{\mathrm{e}}}{dt}$$

$I_C = I$: the current in the wire passes through our Amperian loop.

$\frac{d\phi_{\mathrm{e}}}{dt} = 0$: no electric field passes through our Amperian loop.



What is the magnetic field at position A?

$$\oint_C \vec{B} \cdot d\vec{l} = B 2\pi r = \mu_0 I_C + \mu_0 \epsilon_0 \frac{d\phi_e}{dt}$$

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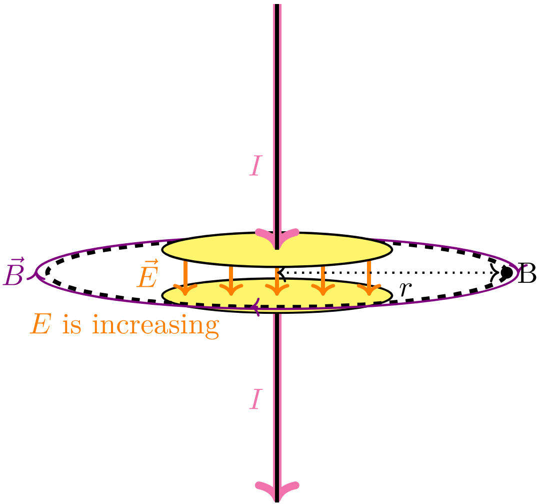
$$B 2\pi r = \mu_0 (I + 0) \quad \text{so } B_{\text{at A}} = \frac{\mu_0 I}{2\pi r}$$

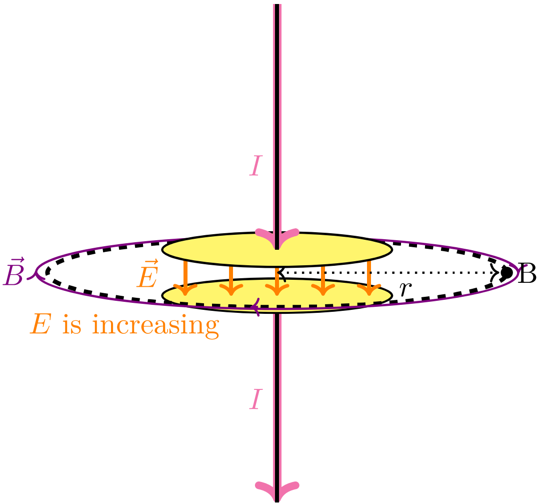


Position B is also r from the wire, but vertically at the center of the capacitor. What is the magnetic field at position B?

$$\oint_{\mathrm{C}} \vec{B} \cdot d\vec{l} = B 2\pi r = \mu_0 I_{\mathrm{C}} + \mu_0 \epsilon_0 \frac{d\phi_{\mathrm{e}}}{dt}$$

$I_C = 0$: no actual current passes through our Amperian loop.

$\frac{d\phi_{\mathrm{e}}}{dt} = \epsilon_0 A \frac{dE}{dt}$: the electric field passes through our Amperian loop and is changing in time.



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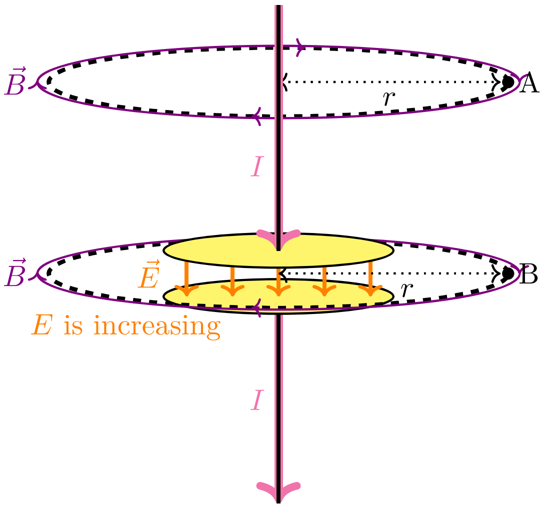
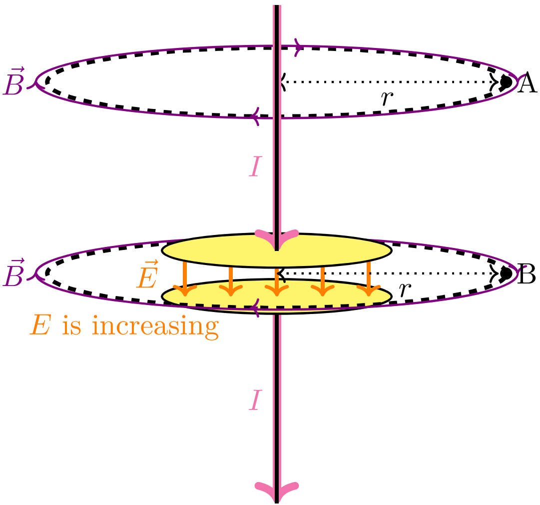
$$B2\pi r = \mu_0 \left(0 + \epsilon_0 A \frac{dE}{dt} \right) \text{ so}$$

$$B_{\text{at B}} = \frac{\mu_0 \epsilon_0 A \frac{dE}{dt}}{2\pi r}$$



$$B_{\text{at A}} = \frac{\mu_0 I}{2\pi r} \text{ and } B_{\text{at B}} = \frac{\mu_0 \epsilon_0 A \dfrac{dE}{dt}}{2\pi r}$$



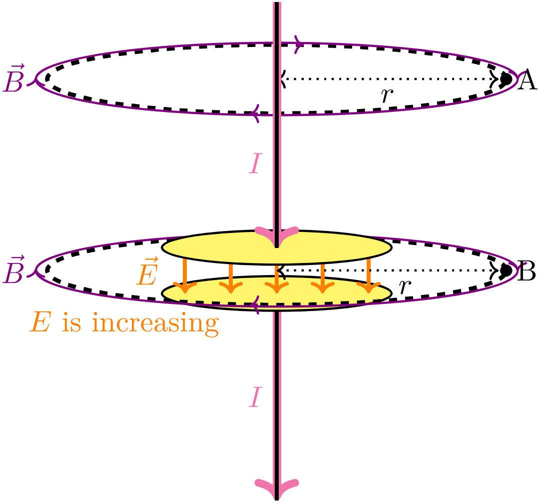
$$B_{\text{at A}} = \frac{\mu_0 I}{2\pi r} \text{ and } B_{\text{at B}} = \frac{\mu_0 \epsilon_0 A \dfrac{dE}{dt}}{2\pi r}$$

E-field inside a capacitor: $E = \dfrac{|\sigma|}{\epsilon_0} = \dfrac{Q}{A\epsilon_0}$

Therefore $\dfrac{dE}{dt} = \dfrac{1}{A\epsilon_0}\dfrac{dQ}{dt} = \dfrac{I}{\epsilon_0 A}$

The displacement current passing through the capacitor is $I_{\text{d}} = \epsilon_0 \dfrac{d\phi_{\text{e}}}{dt} = \epsilon_0 A \dfrac{dE}{dt} = I$

The displacement current passing through the capacitor is equal to the actual current in the wire that is charging the capacitor



$$B_{\text{at A}} = \frac{\mu_0 I}{2\pi r} \text{ and } B_{\text{at B}} = \frac{\mu_0 I_{\text{d}}}{2\pi r}$$

Actual current passes through the Amperian Loop at position A. Displacement current passes through the Amperian loop at position B. But these are the same.

$$B_{\text{at A}} = B_{\text{at B}}$$