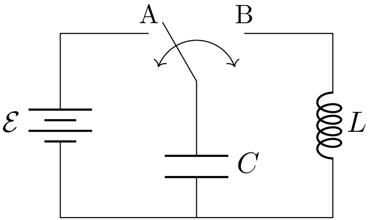
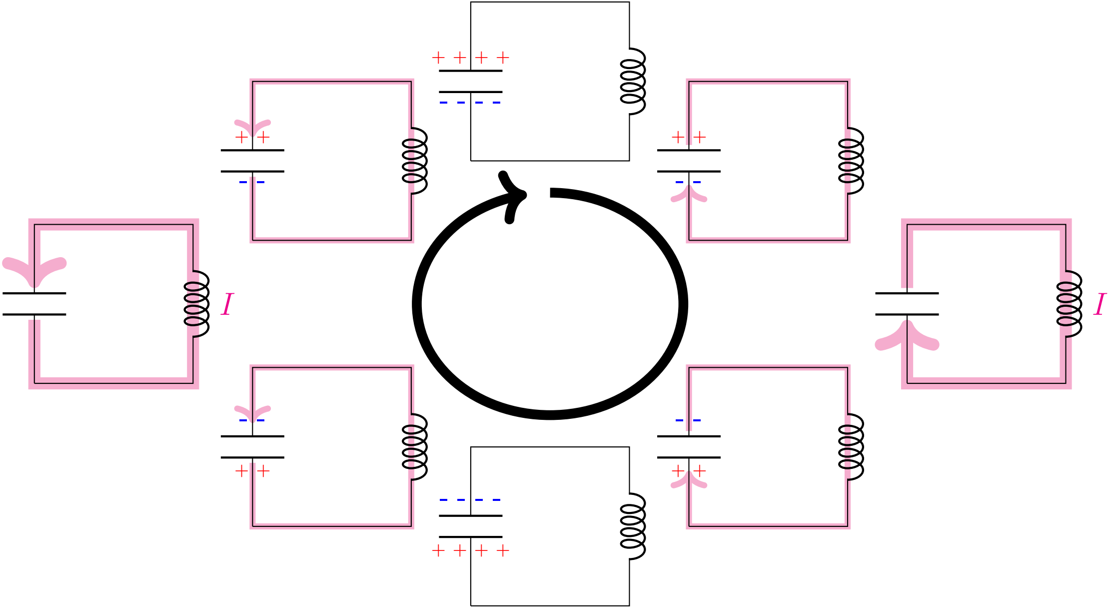
LC Circuits

SP212 - General Physics II

Imagine the circuit shown below. The switch is moved left (to position A). The battery quickly charges the capacitor to $Q_{\text{peak}} = C\mathcal{E}$. Then, at time $t = 0$, the switch is flipped right (to position B). What happens?



1. With the battery removed, the capacitor will discharge. Current will flow clockwise.
2. Eventually, the capacitor is uncharged. But at this point the current is large and the inductor causes the current to keep going. This causes the capacitor to recharge *with the opposite polarity*.
3. Eventually the current falls to zero. At this point the capacitor is fully charged, but with opposite polarity from how it started.
4. Current flows counter-clockwise to discharge the capacitor.
5. When the capacitor is uncharged, the current is large and the inductor keeps the current going.
6. Eventually the current falls to zero when the capacitor is back to where it started. The process repeats.


+ + + +
- - - -
+ +
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I
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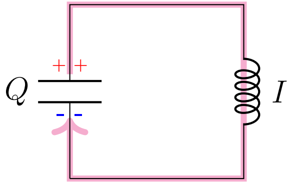
+ + + +

LC circuits

Do a clockwise loop rule. Define Q as positive when the top plate is positive. Define a clockwise current as positive.



$$+\frac{Q}{C} - L\frac{dI}{dt} = 0$$

$I = -\frac{dQ}{dt}$ (positive current leads away from the positive plate).

$$+\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0 \text{ or } \frac{d^2Q}{dt^2} = -\frac{Q}{LC} = -\omega^2 Q \text{ where } \omega = \frac{1}{\sqrt{LC}}$$

Solve this differential equation with initial conditions $Q(t=0) = Q_{\text{peak}}$ and $I(t=0) = 0$:

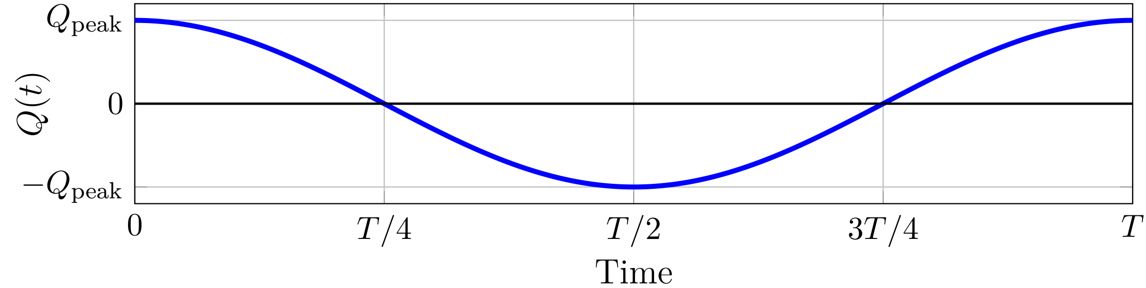
$$Q(t) = Q_{\text{peak}} \cos \omega t \text{ and } I(t) = -\omega Q_{\text{peak}} \sin \omega t$$

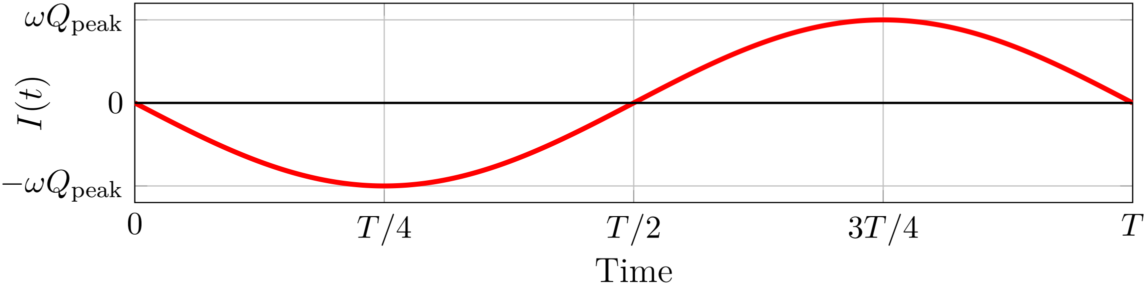
$$\omega = \frac{1}{\sqrt{LC}} = 2\pi f = \frac{2\pi}{T}$$ Angular frequency of the oscillation

$$Q(t) = Q_{\text{peak}} \cos \omega t$$ Charge on the capacitor as a function of time

$$I(t) = -\omega Q_{\text{peak}} \sin \omega t$$ Current through the inductor as a function of time

- C is the capacitance in this circuit [units are farads, F]
- L is the inductance in this circuit [units are henries, H]
- ω is the angular frequency of the oscillations [units are radians per second, rad/s]
  - Note: units are rad/s. Your calculator must be set for radians.
  - f is the frequency [units are hertz, Hz]; T is the period [units are seconds, s]
- Q_{peak} is the maximum (peak) charge on the capacitor. It is assumed that we define $t = 0$ as a moment when this is the charge on the capacitor.


Q_{peak}
$Q(t)$
0
$-Q_{\text{peak}}$
0
$T/4$
$T/2$
$3T/4$
T
Time


ωQ_{peak}
$I(t)$
0
$-\omega Q_{\text{peak}}$
0
$T/4$
$T/2$
$3T/4$
T
Time

Energy in an LC Circuit

Energy stored in the capacitor: $U_{\mathrm{e}} = \dfrac{Q^2}{2C} = \dfrac{[Q_{\mathrm{peak}} \cos \omega t]^2}{2C} = \dfrac{Q_{\mathrm{peak}}^2}{2C} \cos^2 \omega t$

Energy in an LC Circuit

Energy stored in the capacitor: $U_\mathrm{e} = \dfrac{Q^2}{2C} = \dfrac{[Q_\mathrm{peak}\cos\omega t]^2}{2C} = \dfrac{Q_\mathrm{peak}^2}{2C}\cos^2\omega t$

Energy stored in the inductor: $U_\mathrm{m} = \dfrac{1}{2}LI^2 = \dfrac{1}{2}L[-\omega Q_\mathrm{peak}\sin\omega t]^2 = \dfrac{L\omega^2 Q_\mathrm{peak}^2}{2}\sin^2\omega t$

Remember, $\omega = \dfrac{1}{\sqrt{LC}}$ which means $\dfrac{\omega^2 L}{2} = \dfrac{1}{2C}$

This gives $U_\mathrm{m} = \dfrac{Q_\mathrm{peak}^2}{2C}\sin^2\omega t$

Energy in an LC Circuit

Energy stored in the capacitor: $U_\mathrm{e} = \dfrac{Q^2}{2C} = \dfrac{[Q_\mathrm{peak}\cos\omega t]^2}{2C} = \dfrac{Q_\mathrm{peak}^2}{2C}\cos^2\omega t$

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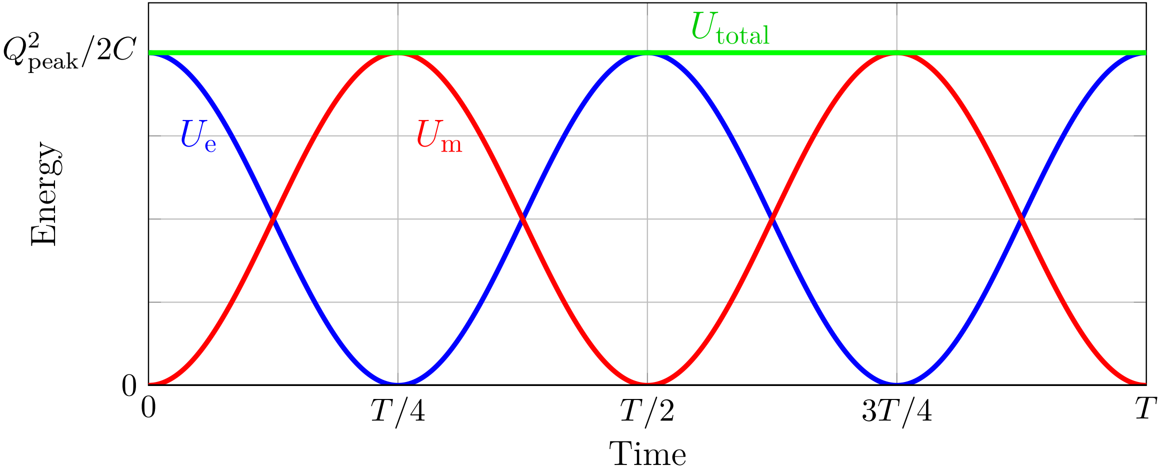
Remember, $\omega = \dfrac{1}{\sqrt{LC}}$ which means $\dfrac{\omega^2 L}{2} = \dfrac{1}{2C}$

This gives $U_\mathrm{m} = \dfrac{Q_\mathrm{peak}^2}{2C}\sin^2\omega t$

<u>Total</u> energy stored: $U = U_\mathrm{e} + U_\mathrm{m} = \dfrac{Q_\mathrm{peak}^2}{2C}\left[\cos^2\omega t + \sin^2\omega t\right] = \dfrac{Q_\mathrm{peak}^2}{2C}$

This is constant in time! This circuit had no resistor to dissipate heat.



Interesting to note: the energies will cycle twice each period. After half of a period the capacitor is charged with opposite polarity, but $U_{\text{e}} \propto Q^2$ so the minus sign doesn't matter.