Angular Momentum Eigenstates

Reminders: $\hat{L}^2|l\ m\rangle = \hbar^2 l(l+1)|l\ m\rangle$, $\quad \hat{L}_z|l\ m\rangle = m\hbar|l\ m\rangle$
$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$, and $\hat{L}_\pm|l\,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|l\,(m\pm 1)\rangle$

$\hat{L}_+|1\ \ -1\rangle = \sqrt{2}\hbar|1\ \ 0\rangle \qquad \hat{L}_+|1\ \ 0\rangle = \sqrt{2}\hbar|1\ \ +1\rangle \qquad \hat{L}_+|1\ \ +1\rangle = 0$
$\hat{L}_-|1\ \ -1\rangle = 0 \qquad \hat{L}_-|1\ \ 0\rangle = \sqrt{2}\hbar|1\ \ -1\rangle \qquad \hat{L}_-|1\ \ +1\rangle = \sqrt{2}\hbar|1\ \ 0\rangle$

Question 1: A particle's state is given by

$$|\Psi\rangle = \frac{1}{3}|1\ \ -1\rangle + \frac{2}{3}|1\ \ 0\rangle + \frac{2i}{3}|1\ \ +1\rangle$$

expressed as a superposition of $|l\ \ m\rangle$ angular momentum eigenstates.

(a) What is $\langle L_z\rangle$?

(b) What is $\langle L_x\rangle$?

(c) What is $\langle L_y\rangle$?

Question 2: A particle's state is given by

$$|\Psi\rangle = \frac{3}{\sqrt{24}}|2 \quad +2\rangle - \frac{1}{\sqrt{24}}|2 \quad +1\rangle + \frac{i}{\sqrt{24}}|2 \;\; 0\rangle - \frac{2}{\sqrt{24}}|1 \quad +1\rangle - \frac{3i}{\sqrt{24}}|1 \;\; 0\rangle$$

expressed as a superposition of $|l \quad m\rangle$ angular momentum eigenstates.

(a) What is $\langle L_z \rangle$?

(b) You measure L^2 and obtain the answer $2\hbar^2$. After that measurement, what is the normalized state of the particle? What is $\langle L_z \rangle$ now?

SOLUTIONS

Question 1:

To start: $\hat{L}_+|1\ \ 1\rangle = 0, \quad \hat{L}_-|1\ \ 1\rangle = \sqrt{2}\hbar|1\ \ 0\rangle$
$\hat{L}_+|1\ \ 0\rangle = \sqrt{2}\hbar|1\ \ 1\rangle, \quad \hat{L}_-|1\ \ 0\rangle = \sqrt{2}\hbar|1\ \ -1\rangle$
$\hat{L}_+|1\ \ -1\rangle = \sqrt{2}\hbar|1\ \ 0\rangle, \quad \hat{L}_-|1\ \ -1\rangle = 0$

(a) $\hat{L}_z|\Psi\rangle = \frac{1}{3}(-\hbar)|1\ \ -1\rangle + \frac{2}{3}(0)|1\ \ 0\rangle + \frac{2i}{3}(\hbar)|1\ \ +1\rangle$
$\langle L_z\rangle = \left|\frac{1}{3}\right|^2(-\hbar) + \left|\frac{2}{3}\right|^2(0) + \left|\frac{2i}{3}\right|^2(\hbar) = \frac{1}{9}(-\hbar) + 0 + \frac{4}{9}\hbar$

$$\boxed{\langle L_z\rangle = \frac{1}{3}\hbar}$$

$\hat{L}_+|\Psi\rangle = \frac{1}{3}\hat{L}_+|1\ \ -1\rangle + \frac{2}{3}\hat{L}_+|1\ \ 0\rangle + \frac{2i}{3}\hat{L}_+|1\ \ +1\rangle = \frac{1}{3}\sqrt{2}\hbar|1\ 0\rangle + \frac{2}{3}\sqrt{2}\hbar|1\ 1\rangle$
$\hat{L}_-|\Psi\rangle = \frac{1}{3}\hat{L}_-|1\ \ -1\rangle + \frac{2}{3}\hat{L}_-|1\ \ 0\rangle + \frac{2i}{3}\hat{L}_-|1\ \ +1\rangle = \frac{2}{3}\sqrt{2}\hbar|1\ \ -1\rangle + \frac{2i}{3}\sqrt{2}\hbar|1\ 0\rangle$

$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$ so $\hat{L}_x|\Psi\rangle = \frac{\sqrt{2}\hbar}{3}|1\ \ -1\rangle + \frac{\sqrt{2}\hbar(1+2i)}{6}|1\ 0\rangle + \frac{\sqrt{2}\hbar}{3}|1\ 1\rangle$

$\langle L_x\rangle = \frac{1}{3}\cdot\frac{\sqrt{2}\hbar}{3} + \frac{2}{3}\cdot\frac{\sqrt{2}\hbar(1+2i)}{6} + (\frac{2i}{3})^*\cdot\frac{\sqrt{2}\hbar}{3}$ so $\boxed{\langle L_x\rangle = \frac{2\sqrt{2}\hbar}{9} \approx 0.314\hbar}$

$\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$ so $\hat{L}_y|\Psi\rangle = \frac{-\sqrt{2}\hbar}{3i}|1\ \ -1\rangle + \frac{\sqrt{2}\hbar(1-2i)}{6i}|1\ 0\rangle + \frac{\sqrt{2}\hbar}{3i}|1\ 1\rangle$

$\langle L_y\rangle = \frac{1}{3}\cdot\frac{-\sqrt{2}\hbar}{3i} + \frac{2}{3}\cdot\frac{\sqrt{2}\hbar(1-2i)}{6i} + (\frac{2i}{3})^*\cdot\frac{\sqrt{2}\hbar}{3i}$ so $\boxed{\langle L_y\rangle = \frac{-4\sqrt{2}\hbar}{9} \approx -0.629\hbar}$

Question 2:

(a) $\langle L_z\rangle = \frac{9}{24}(2\hbar) + \frac{1}{24}(1\hbar) + \frac{1}{24}(0) + \frac{4}{24}(1\hbar) + \frac{9}{24}(0)$ so $\boxed{\langle L_z\rangle = \frac{23}{24}\hbar}$

(b) Measuring $L^2 = 2\hbar^2$ collapses the quantum state to the superposition of the two states with $l = 1$: $-\frac{2}{\sqrt{24}}|1\ \ +1\rangle - \frac{3i}{\sqrt{24}}|1\ \ 0\rangle$. Renormalizing, this gives $|\Psi\rangle = -\frac{2}{\sqrt{13}}|1\ \ +1\rangle - \frac{3i}{\sqrt{13}}|1\ \ 0\rangle$. $\langle L_z\rangle = \frac{4}{13}(1\hbar) + \frac{9}{13}(0)$ so $\boxed{\langle L_z\rangle = \frac{4}{13}\hbar}$