Angular Momentum Eigenfunctions

A particle is confined to the surface of a sphere. Its wave function is given by

$$\Psi(\theta,\,\phi) = \begin{cases} \sqrt{\dfrac{1}{2\pi}} & \theta \le \dfrac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

meaning that the particle is confined to the northern hemisphere but its wave function is uniform over that hemisphere. You measure L^2; what is the probability that the answer will be $2\hbar^2$?

Note: the three spherical harmonics with $l = 1$ are $Y_1^0 = \sqrt{\dfrac{3}{4\pi}}\cos\theta$, and $Y_1^{\pm 1} = \mp\sqrt{\dfrac{3}{8\pi}}\sin\theta\, e^{\mp i\phi}$. $d\Omega = \sin\theta\, d\theta\, d\phi$. Always do the ϕ integral first: for two of the three this will be zero. Also, $\displaystyle\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta = \frac{1}{2}$.

Solution:

A $\hat{L}^2$ eigenvalue of $2\hbar^2$ corresponds to $l = 1$. There are three states with $l = 1$: $m = -1,\ 0,\ +1$. The probability of measuring $2\hbar^2$ is $P_{2\hbar^2} = |c_{1,-1}|^2 + |c_{1,0}|^2 + |c_{1,+1}|^2$ where $c_{1,m} = \langle 1\ m|\Psi\rangle = \displaystyle\int (Y_1^m)^*\,\Psi\,d\Omega = \int_0^{2\pi}\int_0^{\pi} (Y_1^m)^*\,\Psi \sin\theta\,d\theta\,d\phi$.

$\displaystyle\int_0^{2\pi} e^{im\phi} d\phi = 0$ for any nonzero integer m (and 2π when $m = 0$). This wave function had no ϕ-dependence, therefore $c_{1,\pm 1} = 0$.

$$c_{1,0} = \int_0^{2\pi} d\phi \int_0^{\pi/2} \left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)^* \sqrt{\frac{1}{2\pi}}\sin\theta\,d\theta$$

where I've done the integral only up to $\theta = \dfrac{\pi}{2}$ because Ψ is 0 at higher angles.

$$c_{1,0} = 2\pi\sqrt{\frac{3}{8\pi^2}}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = 2\pi\sqrt{\frac{3}{8\pi^2}}\frac{1}{2} = \sqrt{\frac{3}{8}}$$

$\boxed{\text{The probability of measuring } 2\hbar^2 \text{ is } P_{2\hbar^2} = \frac{3}{8} = 0.375}$