Spin

The $\hat{S}_y$ matrix is given by

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(a) What are the two eigenspinors of $\hat{S}_y$? i.e., what are $\chi_+^{(y)}$ and $\chi_-^{(y)}$?

The spin state of a spin-$\frac{1}{2}$ particle is given by $\chi = \frac{1}{\sqrt{15}}\begin{pmatrix} 2+i \\ 1-3i \end{pmatrix}$

(b) You measure S_y. What is the probability of measuring $+\frac{\hbar}{2}$?

(c) What is $\langle S_y \rangle$?

Solutions:

(a) Eigenspinors of $\hat{S}_y$

The $\hat{S}_y$ matrix is given by:

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1}$$

To find the eigenspinors, we solve the eigenvalue equation $\hat{S}_y\chi = \lambda\chi$. The eigenvalues for a spin-1/2 system are $\lambda = \pm\frac{\hbar}{2}$.

For the $+\frac{\hbar}{2}$ state ($\chi_+^{(y)}$):

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$
$$-ib = a \implies b = ia$$

Normalizing the spinor ($|a|^2 + |b|^2 = 1$), we get $a = \frac{1}{\sqrt{2}}$. Thus,

$$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{2}$$

For the $-\frac{\hbar}{2}$ state ($\chi_-^{(y)}$):

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$
$$-ib = -a \implies b = -ia$$

Normalizing this spinor gives:

$$\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{3}$$

(b) Probability of measuring $+\frac{\hbar}{2}$

The initial spin state is given by:

$$\chi = \frac{1}{\sqrt{15}}\begin{pmatrix} 2+i \\ 1-3i \end{pmatrix} \tag{4}$$

The probability of measuring $+\frac{\hbar}{2}$ is $P(+) = |\langle\chi_+^{(y)}|\chi\rangle|^2$.

First, we calculate the inner product:

$$\begin{aligned}\langle\chi_+^{(y)}|\chi\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{15}}\begin{pmatrix}2+i\\1-3i\end{pmatrix}\\ &= \frac{1}{\sqrt{30}}[1(2+i)-i(1-3i)]\\ &= \frac{1}{\sqrt{30}}(2+i-i+3i^2)\\ &= \frac{1}{\sqrt{30}}(2-3) = -\frac{1}{\sqrt{30}}\end{aligned}$$

Squaring the amplitude gives the probability:

$$P(+) = \left|-\frac{1}{\sqrt{30}}\right|^2 = \frac{1}{30} \tag{5}$$

(c) Expectation Value $\langle S_y\rangle$

We can calculate the expectation value using $\langle S_y\rangle = P(+)\left(+\frac{\hbar}{2}\right) + P(-)\left(-\frac{\hbar}{2}\right)$. Since total probability must sum to 1:

$$P(-) = 1 - P(+) = 1 - \frac{1}{30} = \frac{29}{30} \tag{6}$$

Now, compute the expectation value:

$$\begin{aligned}\langle S_y\rangle &= \left(\frac{1}{30}\right)\left(\frac{\hbar}{2}\right) + \left(\frac{29}{30}\right)\left(-\frac{\hbar}{2}\right)\\ &= \frac{\hbar}{60} - \frac{29\hbar}{60}\\ &= -\frac{28\hbar}{60}\\ &= -\frac{7\hbar}{15}\end{aligned}$$

Alternative Method using Matrix Multiplication:

$$
\begin{aligned}
\langle S_y \rangle &= \chi^\dagger \hat{S}_y \chi \\
&= \frac{1}{\sqrt{15}} \begin{pmatrix} 2-i & 1+3i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{15}} \begin{pmatrix} 2+i \\ 1-3i \end{pmatrix} \\
&= \frac{\hbar}{30} \begin{pmatrix} 2-i & 1+3i \end{pmatrix} \begin{pmatrix} -3-i \\ -1+2i \end{pmatrix} \\
&= \frac{\hbar}{30}[(2-i)(-3-i) + (1+3i)(-1+2i)] \\
&= \frac{\hbar}{30}[(-7+i) + (-7-i)] \\
&= \frac{\hbar}{30}(-14) = -\frac{7\hbar}{15}
\end{aligned}
$$