Addition of Angular Momentum

$$s = 0 \text{ (singlet)} \qquad |0\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$s = 1 \text{ (triplet)} \qquad \begin{cases} |1\ +1\rangle = |\uparrow\uparrow\rangle \\ |1\ 0\rangle = \dfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1\ -1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

Also recall that $\hat{S}^2|s\ m\rangle = \hbar^2 s(s+1)|s\ m\rangle$ and $\hat{S}_z|s\ m\rangle = m\hbar|s\ m\rangle$.

You have a system of two spin-1/2 particles. The quantum state vector of your system is given by $|\Psi\rangle = \dfrac{1}{\sqrt{10}}(|\uparrow\uparrow\rangle + 2i|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - 2|\downarrow\downarrow\rangle)$.

(a) You measure S_z, the z-component of the total spin of the two-particle system. What are the possible values you might measure and their probabilities? What is its expectation value?

(b) Instead you measure S^2, the square of the total spin of the two-particle system. What are the possible values you might measure and their probabilities? What is its expectation value?

SOLUTIONS

(a) There are of course a few ways to do this, but we can simply note that the problem explicitly gives the quantum state vector expanded as a superposition of four states that are each eigenstates of the $\hat{S}_z$ operator. That makes it pretty straightforward.

$\boxed{\left|\frac{1}{\sqrt{10}}\right|^2 = 0.1}$ probability of measuring $S_z = +\hbar$

$\boxed{\left|\frac{2i}{\sqrt{10}}\right|^2 + \left|\frac{-1}{\sqrt{10}}\right|^2 = 0.5}$ probability of measuring $S_z = 0$

$\boxed{\left|\frac{-2}{\sqrt{10}}\right|^2 = 0.4}$ probability of measuring $S_z = -\hbar$

The expectation value is $\boxed{\langle S_z\rangle = 0.1(+\hbar) + 0.5(0) + 0.4(-\hbar) = -0.3\hbar}$

(b) If we want to know the probability of various outcomes of a S^2 measurement we would like to write the quantum state in terms of eigenstates of the $\hat{S}^2$ operator (the singlet and triplet states). We would like to write

$$|\Psi\rangle = c_{0,0}|0\ 0\rangle + c_{1,1}|1\ 1\rangle + c_{1,0}|1\ 0\rangle + c_{1,-1}|1\ -1\rangle$$

where the coefficients can be determined as $c_{0,0} = \langle 0\ 0|\Psi\rangle$, etc.

$c_{0,0} = \frac{1}{\sqrt{2}}(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)\frac{1}{\sqrt{10}}(|\uparrow\uparrow\rangle + 2i|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - 2|\downarrow\downarrow\rangle) = \frac{1}{\sqrt{20}}(2i+1)$.

Similarly,

$c_{1,1} = \langle\uparrow\uparrow|\frac{1}{\sqrt{10}}(|\uparrow\uparrow\rangle + 2i|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - 2|\downarrow\downarrow\rangle) = \frac{1}{\sqrt{10}}$,

$c_{1,0} = \frac{1}{\sqrt{2}}(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)\frac{1}{\sqrt{10}}(|\uparrow\uparrow\rangle + 2i|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - 2|\downarrow\downarrow\rangle) = \frac{1}{\sqrt{20}}(2i-1)$, and

$c_{1,1} = \langle\downarrow\downarrow|\frac{1}{\sqrt{10}}(|\uparrow\uparrow\rangle + 2i|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - 2|\downarrow\downarrow\rangle) = \frac{-2}{\sqrt{10}}$.

This gives $|\Psi\rangle = \frac{2i+1}{\sqrt{20}}|0\ 0\rangle + \frac{1}{\sqrt{10}}|1\ 1\rangle + \frac{2i-1}{\sqrt{20}}|1\ 0\rangle + \frac{-2}{\sqrt{10}}|1\ -1\rangle$

$\boxed{\left|\frac{2i+1}{\sqrt{20}}\right|^2 = 0.25}$ probability of measuring $S^2 = 0$

$\boxed{\left|\frac{1}{\sqrt{10}}\right|^2 + \left|\frac{2i-1}{\sqrt{20}}\right|^2 + \left|\frac{-2}{\sqrt{10}}\right|^2 = 0.1 + 0.25 + 0.4 = 0.75}$

probability of measuring $S^2 = 2\hbar^2$

The expectation value is $\boxed{\langle S^2\rangle = 0.25(0) + 0.75(2\hbar^2) = 1.5\hbar^2}$