Clebsch-Gordan Tables

The Clebsch-Gordan tables can be found on page 179 of your textbook.

Question 1: Suppose you have a $s_1 = 3/2$ particle and a $s_2 = 1$ particle.

(a) They are in the $\left|\frac{5}{2}\ \frac{+1}{2}\right\rangle$ state. Write this state as an expansion in terms of $|\frac{3}{2}\ m_1\rangle|1\ m_2\rangle$ states.

(b) Suppose instead they are in the $\left|\frac{3}{2}\ \frac{+1}{2}\right\rangle$ state. Write this state as an expansion in terms of $|\frac{3}{2}\ m_1\rangle|1\ m_2\rangle$ states.

(c) Take the inner product of the two expansions you wrote out above. Do so to explicitly confirm that $\left\langle\frac{5}{2}\ \frac{+1}{2}\middle|\frac{3}{2}\ \frac{+1}{2}\right\rangle = 0$.

Question 2: Now you have two spin-1 particles.

(a) These particles are in the state $|1\ +1\rangle|1\ -1\rangle$. Suppose you were to measure S^2 (the total spin squared). What are the possible measured values and their probabilities?

(b) Suppose you perform the measurement and obtain $S^2 = 6\hbar^2$ as your answer. Then you measure $S_z^{(1)}$, the z-component of the spin of particle #1 (where particle #1 is the one that originally had $m = +1$ or $S_z = +\hbar$ - this assumes that the two particles are somehow distinguishable). What are the possible measured values and their probabilities?

SOLUTIONS

Question 1:

(a) $\boxed{\left|\frac{5}{2}\ \frac{+1}{2}\right\rangle = \sqrt{\frac{1}{10}}\left|\frac{3}{2}\ \frac{+3}{2}\right\rangle\left|1\ -1\right\rangle + \sqrt{\frac{3}{5}}\left|\frac{3}{2}\ \frac{+1}{2}\right\rangle\left|1\ 0\right\rangle + \sqrt{\frac{3}{10}}\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle\left|1\ +1\right\rangle}$

(b) $\boxed{\left|\frac{3}{2}\ \frac{+1}{2}\right\rangle = \sqrt{\frac{2}{5}}\left|\frac{3}{2}\ \frac{+3}{2}\right\rangle\left|1\ -1\right\rangle + \sqrt{\frac{1}{15}}\left|\frac{3}{2}\ \frac{+1}{2}\right\rangle\left|1\ 0\right\rangle - \sqrt{\frac{8}{15}}\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle\left|1\ +1\right\rangle}$

(c) $\left\langle\frac{5}{2}\ \frac{+1}{2}\middle|\frac{3}{2}\ \frac{+1}{2}\right\rangle = \sqrt{\frac{1}{10}}\sqrt{\frac{2}{5}} + \sqrt{\frac{3}{5}}\sqrt{\frac{1}{15}} - \sqrt{\frac{3}{10}}\sqrt{\frac{8}{15}}$

$$\boxed{\left\langle\frac{5}{2}\ \frac{+1}{2}\middle|\frac{3}{2}\ \frac{+1}{2}\right\rangle = \sqrt{\frac{2}{50}} + \sqrt{\frac{3}{75}} - \sqrt{\frac{24}{150}} = \sqrt{\frac{3}{75}} + \sqrt{\frac{3}{75}} - 2\sqrt{\frac{3}{75}} = 0}$$

Question 2:

$|1\ +1\rangle|1\ -1\rangle = \sqrt{\frac{1}{6}}|2\ 0\rangle + \sqrt{\frac{1}{2}}|1\ 0\rangle + \sqrt{\frac{1}{3}}|0\ 0\rangle$

(a) $\frac{1}{6}$ probability of measuring $S^2 = \hbar^2 2(2+1) = 6\hbar^2$, $\frac{1}{2}$ probability of measuring $S^2 = \hbar^2 1(1+1) = 2\hbar^2$, $\frac{1}{3}$ probability of measuring $S^2 = \hbar^2 0(0+1) = 0$.

(b) Measuring $S^2 = 6\hbar^2$ collapses the state to $|2\ 0\rangle$. Using the chart,

$$|2\ 0\rangle = \sqrt{\frac{1}{6}}|1\ +1\rangle|1\ -1\rangle + \sqrt{\frac{2}{3}}|1\ 0\rangle|1\ 0\rangle + \sqrt{\frac{1}{6}}|1\ -1\rangle|1\ +1\rangle$$

So, $\frac{1}{6}$ probability of measuring $S_z^{(1)} = +\hbar$, $\frac{2}{3}$ probability of measuring $S_z^{(1)} = 0$, $\frac{1}{6}$ probability of measuring $S_z^{(1)} = -\hbar$