

Nondegenerate Perturbation Theory

Suppose we have a simple harmonic oscillator that has a delta-function barrier in the middle: $V(x) = \frac{1}{2}m\omega^2 x^2 + \alpha\delta(x)$.

(α is a constant that describes the strength of the delta-function. It is positive for a barrier. It has dimensions of energy times length.)

(a) What is your estimate of the ground state energy? Do first order perturbation theory plus second order theory, keeping the first two non-zero terms in the second-order expansion.

(b) What is your estimate of the ground state wave function? Do first order perturbation theory, keeping the first two non-zero terms in the expansion.

(c) Qualitatively, how is the first excited state ($\psi_1(x)$) affected by the perturbation?

$$E_n^1 = \langle \psi_n^0 | \hat{H}' | \psi_n^0 \rangle \quad \psi_n^1 = \sum_{m \neq n} \frac{\langle \psi_m^0 | \hat{H}' | \psi_n^0 \rangle}{E_n^0 - E_m^0} \psi_m^0 \quad E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | \hat{H}' | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$E_n^0 = \hbar\omega \left(n + \frac{1}{2} \right) \quad \text{Note in the below: } \xi \equiv \sqrt{\frac{m\omega}{\hbar}} x$$

$$\begin{aligned} \psi_0^0(\xi) &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\xi^2/2} & \psi_1^0(\xi) &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \sqrt{2}\xi e^{-\xi^2/2} \\ \psi_2^0(\xi) &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{2}}(2\xi^2 - 1)e^{-\xi^2/2} & \psi_3^0(\xi) &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{3}}(2\xi^3 - 3\xi)e^{-\xi^2/2} \\ \psi_4^0(\xi) &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{24}}(4\xi^4 - 12\xi^2 + 3)e^{-\xi^2/2} \end{aligned}$$

Note: $\langle \psi_m^0 | \hat{H}' | \psi_n^0 \rangle = \int [\psi_m^0]^* \alpha \delta(x) \psi_n^0 dx = \alpha [\psi_m^0]^* \psi_n^0 \Big|_{x=0}$

$$\langle \psi_0^0 | \hat{H}' | \psi_0^0 \rangle = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \alpha \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} = \alpha \sqrt{\frac{m\omega}{\pi\hbar}}$$

$$\langle \psi_1^0 | \hat{H}' | \psi_0^0 \rangle = 0 \cdot \alpha \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} = 0$$

$$\langle \psi_2^0 | \hat{H}' | \psi_0^0 \rangle = \frac{-1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \alpha \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} = -\alpha \sqrt{\frac{m\omega}{2\pi\hbar}}$$

$$\langle \psi_3^0 | \hat{H}' | \psi_0^0 \rangle = 0 \cdot \alpha \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} = 0$$

$$\langle \psi_4^0 | \hat{H}' | \psi_0^0 \rangle = \frac{3}{\sqrt{24}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \alpha \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} = \alpha \sqrt{\frac{9m\omega}{24\pi\hbar}}$$

$$(a) E_0^1 = \alpha \sqrt{\frac{m\omega}{\pi\hbar}}$$

$$E_0^2 = \frac{|0|^2}{0.5\hbar\omega - 1.5\hbar\omega} + \frac{\left| -\alpha \sqrt{\frac{m\omega}{2\pi\hbar}} \right|^2}{0.5\hbar\omega - 2.5\hbar\omega} + \frac{|0|^2}{0.5\hbar\omega - 3.5\hbar\omega} + \frac{\left| \alpha \sqrt{\frac{9m\omega}{24\pi\hbar}} \right|^2}{0.5\hbar\omega - 4.5\hbar\omega} + \dots$$

$$E_0^2 \approx -\frac{m\alpha^2}{4\pi\hbar^2} - \frac{9m\alpha^2}{96\pi\hbar^2}$$

$$E_0 \approx \frac{1}{2}\hbar\omega + \alpha \sqrt{\frac{m\omega}{\pi\hbar}} - \frac{m\alpha^2}{4\pi\hbar^2} - \frac{9m\alpha^2}{96\pi\hbar^2}$$

$$(b) \psi_1^1 = 0\psi_1^0 + \frac{-\alpha \sqrt{\frac{m\omega}{2\pi\hbar}}}{0.5\hbar\omega - 2.5\hbar\omega} \psi_2^0 + 0\psi_3^0 + \frac{\alpha \sqrt{\frac{9m\omega}{24\pi\hbar}}}{0.5\hbar\omega - 4.5\hbar\omega} \psi_4^0 + \dots$$

$$\psi_1^1 \approx \alpha \sqrt{\frac{m}{8\pi\omega\hbar^3}} \psi_2^0 - \alpha \sqrt{\frac{9m}{364\pi\omega\hbar^3}} \psi_4^0$$

$$\psi_0 \approx \psi_0^0 + \alpha \sqrt{\frac{m}{8\pi\omega\hbar^3}} \psi_2^0 - \alpha \sqrt{\frac{9m}{364\pi\omega\hbar^3}} \psi_4^0$$

(c) $\psi_1^0(0) = 0$. Therefore ψ_1 (and all ψ_n with odd n) is completely unaffected by this perturbation.