

[15 points] Question #1: A particle is constrained to the surface of a sphere. Its wave function at some moment is given by

$$\Psi(\theta, \phi) = \sqrt{\frac{5}{4\pi}} \cos^2 \theta$$

You measure L^2 , the orbital angular momentum squared of this particle. What is the probability that your answer will be 0 (corresponding to $l = 0$)? (As a reminder, the solid angle differential is $d\Omega = \sin \theta d\theta d\phi$.)

$$\begin{aligned}
 Y_0^0 &= \sqrt{\frac{1}{4\pi}} \Rightarrow \text{the only spherical harmonic with } l=0 \\
 c_{0,0} &= \langle Y_0^0 | \Psi \rangle = \int_0^{2\pi} d\phi \int_0^\pi \left(\frac{1}{\sqrt{4\pi}}\right)^* \sqrt{\frac{5}{4\pi}} \cos^2 \theta \sin \theta d\theta \\
 &= 2\pi \frac{\sqrt{5}}{4\pi} \int_0^\pi \cos^2 \theta \sin \theta d\theta = \frac{\sqrt{5}}{2} \left(-\frac{1}{3} \cos^3 \theta \Big|_0^\pi \right) = \frac{\sqrt{5}}{2} \left(\frac{2}{3} \right) = \frac{\sqrt{5}}{3}
 \end{aligned}$$

$$P_{l=0} = |c_{0,0}|^2 = \left| \frac{\sqrt{5}}{3} \right|^2$$

$$\boxed{\text{Prob. is } \frac{5}{9}}$$

[15 points] Question #2: Express the commutator $[\hat{L}_+, \hat{L}_-]$ as a single operator multiplied by constants.

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

$$[\hat{L}_x + i\hat{L}_y, \hat{L}_x - i\hat{L}_y] \quad [\hat{L}_y, \hat{L}_x] = -i\hbar \hat{L}_z$$

$$= [\hat{L}_x, \hat{L}_x] + i[\hat{L}_y, \hat{L}_x] - i[\hat{L}_x, \hat{L}_y] + [\hat{L}_y, \hat{L}_y]$$

$$= 0 + i(-i\hbar \hat{L}_z) - i(i\hbar \hat{L}_z) + 0 = 2\hbar \hat{L}_z$$

$$[\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z$$

[20 points] Question #3: At some moment in time the state of a particle is given by

$$|\Psi\rangle = \frac{3}{5}|1\ 0\rangle + \frac{4i}{5}|1\ +1\rangle$$

What is $\langle L_y \rangle$ at this moment in time?

Hint: $\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$. The effects of $\hat{L}_+$ and $\hat{L}_-$ on the $|1\ m\rangle$ states are given on the last page of this exam.

$$\hat{L}_+ |\Psi\rangle = \frac{3}{5}\sqrt{2}\hbar |1\ +1\rangle + 0$$

$$\hat{L}_- |\Psi\rangle = \frac{3}{5}\sqrt{2}\hbar |1\ -1\rangle + \frac{4i}{5}\sqrt{2}\hbar |1\ 0\rangle$$

$$\text{so } \hat{L}_y |\Psi\rangle = \frac{1}{2i} \left[-\frac{3\sqrt{2}\hbar}{5} |1\ -1\rangle - \frac{4i\sqrt{2}\hbar}{5} |1\ 0\rangle + \frac{3\sqrt{2}\hbar}{5} |1\ +1\rangle \right]$$

$$\langle L_y \rangle = \left(\frac{3}{5} \langle 1\ 0| - \frac{4i}{5} \langle 1\ +1| \right) \left(-\frac{3\sqrt{2}\hbar}{10i} |1\ -1\rangle - \frac{4i\sqrt{2}\hbar}{10i} |1\ 0\rangle + \frac{3\sqrt{2}\hbar}{10i} |1\ +1\rangle \right)$$

$$= \frac{3}{5} \left(-\frac{4i\sqrt{2}\hbar}{10i} \right) + \left(-\frac{4i}{5} \right) \left(\frac{3\sqrt{2}\hbar}{10i} \right) = -\frac{24\sqrt{2}\hbar}{50}$$

$$\boxed{\langle L_y \rangle = -\frac{12\sqrt{2}}{25}\hbar}$$

[15 points] Question #4: You have a spin- $\frac{1}{2}$ particle in the state

$$\chi = \frac{1}{3} \begin{pmatrix} 2 \\ 1+2i \end{pmatrix}.$$

What is the expectation value $\langle S_x \rangle$?

$$\langle S_x \rangle = \frac{1}{3} \begin{pmatrix} 2 & 1-2i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 \\ 1+2i \end{pmatrix}$$

$$= \frac{\hbar}{18} \begin{pmatrix} 2 & 1-2i \end{pmatrix} \begin{pmatrix} 1+2i \\ 2 \end{pmatrix} = \frac{\hbar}{18} \left[2(1+2i) + (1-2i)2 \right] = \frac{4\hbar}{18}$$

$$\boxed{\langle S_x \rangle = \frac{2}{9} \hbar}$$

[20 points] Question #5: You have two spin-1/2 particles in the state

$$|\Psi\rangle = \frac{1}{3} \left(|\uparrow\downarrow\rangle - 2i|\downarrow\uparrow\rangle + 2|\uparrow\uparrow\rangle \right)$$

You make a measurement of S^2 (the total spin angular momentum magnitude squared). What is the probability that you will measure 0 (corresponding to $s = 0$)?

$s=0$ is the singlet $|0\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

$$\begin{aligned} \langle 0\ 0 | \Psi \rangle &= \left(\frac{1}{\sqrt{2}} \langle \uparrow\downarrow| - \langle \downarrow\uparrow| \right) \frac{1}{3} (|\uparrow\downarrow\rangle - 2i|\downarrow\uparrow\rangle + 2|\uparrow\uparrow\rangle) \\ &= \frac{1}{3\sqrt{2}} (1 + 2i) \end{aligned}$$

$$P_{s=0} = \left| \frac{1+2i}{3\sqrt{2}} \right|^2 = \frac{(1+2i)(1-2i)}{18} = \frac{5}{18}$$

Probability = $\frac{5}{18}$

[15 points] Question 6: You have a spin-3/2 particle coupled with a spin-1 particle. They are in the state $|s = 3/2 \ m = +1/2\rangle$. You chose to measure the z -component of the spin of just the spin-3/2 particle. What are the possible results and their probabilities.

Clebsch-Gordan Table

$$|\frac{3}{2} \ +\frac{1}{2}\rangle = \sqrt{\frac{2}{5}} |\frac{3}{2} \ +\frac{3}{2}\rangle |1 \ -1\rangle + \sqrt{\frac{1}{15}} |\frac{3}{2} \ +\frac{1}{2}\rangle |1 \ 0\rangle - \sqrt{\frac{8}{15}} |\frac{3}{2} \ -\frac{1}{2}\rangle |1 \ +1\rangle$$

$\frac{2}{5}$	probability	$+\frac{3\hbar}{2}$
$\frac{1}{15}$	probability	$+\frac{\hbar}{2}$
$\frac{8}{15}$	probability	$-\frac{\hbar}{2}$