Question #1:

(a) $[\hat{L}^2,\, \hat{L}_z] = [\hat{L}_x^2,\, \hat{L}_z] + [\hat{L}_y^2,\, \hat{L}_z] + [\hat{L}_z^2,\, \hat{L}_z]$

$[\hat{L}_x^2,\, \hat{L}_z] = \hat{L}_x[\hat{L}_x,\, \hat{L}_z] + [\hat{L}_x,\, \hat{L}_z]\hat{L}_x$. $[\hat{L}_x,\, \hat{L}_z] = -i\hbar\hat{L}_y$ so therefore $[\hat{L}_x^2,\, \hat{L}_z] = -i\hbar(\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x)$.

$[\hat{L}_y^2,\, \hat{L}_z] = \hat{L}_y[\hat{L}_y,\, \hat{L}_z] + [\hat{L}_y,\, \hat{L}_z]\hat{L}_y$. $[\hat{L}_y,\, \hat{L}_z] = i\hbar\hat{L}_x$ so therefore $[\hat{L}_y^2,\, \hat{L}_z] = i\hbar(\hat{L}_y\hat{L}_x + \hat{L}_x\hat{L}_y)$.

$[\hat{L}^2,\, \hat{L}_z] = -i\hbar(\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x) + i\hbar(\hat{L}_y\hat{L}_x + \hat{L}_x\hat{L}_y) + 0 = 0$ and therefore $\boxed{[\hat{L}^2,\, \hat{L}_z] = 0}$

(b) $[\hat{L}_\pm,\, \hat{L}_z] = [\hat{L}_x \pm i\hat{L}_y,\, \hat{L}_z] = [\hat{L}_x,\, \hat{L}_z] \pm i[\hat{L}_y,\, \hat{L}_z]$

$[\hat{L}_x,\, \hat{L}_z] = -i\hbar\hat{L}_y$ and $[\hat{L}_y,\, \hat{L}_z] = +i\hbar\hat{L}_x$

Therefore $[\hat{L}_\pm,\, \hat{L}_z] = -i\hbar\hat{L}_y \pm i(i\hbar\hat{L}_x)$

$\pm i^2\hbar\hat{L}_x$ is the same as $\mp\hbar\hat{L}_x$ so $[\hat{L}_\pm,\, \hat{L}_z] = -i\hbar\hat{L}_y \mp \hbar\hat{L}_x$

We can write a -1 as $-1 = \pm\mp$ (i.e those factors always give a negative). So we can then factor a $\mp$ out front and get $[\hat{L}_\pm,\, \hat{L}_z] = \mp(\pm\hbar\hat{L}_y + \hbar\hat{L}_x)$

$\boxed{[\hat{L}_\pm,\, \hat{L}_z] = \mp\hbar(\hat{L}_x \pm \hat{L}_y) = \mp\hbar\hat{L}_\pm}$

Question #2:

(a) $P_{0\hbar^2} = |c_{0,0}|^2$ where $c_{0,0} = \langle Y_0^0|\Psi\rangle$

$$\begin{aligned} c_{0,0} &= \int_0^{2\pi}\int_0^{\pi} \frac{1}{\sqrt{4\pi}}\cdot\sqrt{\frac{3}{8\pi}}\sin\theta\,\sin\theta\,d\theta\,d\phi \\ &= 2\pi\sqrt{\frac{3}{32\pi^2}}\int_0^{\pi}\sin^2\theta\,d\theta = 2\pi\sqrt{\frac{3}{32\pi^2}}\cdot\frac{\pi}{2} \\ &= \sqrt{\frac{3\pi^2}{32}} \end{aligned}$$

$$\boxed{P_{0\hbar^2} = \frac{3\pi^2}{32} \approx 0.9253}$$

(b) $P_{2\hbar^2} = |c_{1,0}|^2 + |c_{1,+1}|^2 + |c_{1,-1}|^2$. We know $c_{1,\pm 1} = 0$ because there is no ϕ dependence in Ψ.

$$\begin{aligned} c_{1,0} &= \int_0^{2\pi}\int_0^{\pi}\sqrt{\frac{3}{4\pi}}\cos\theta\cdot\sqrt{\frac{3}{8\pi}}\sin\theta\,\sin\theta\,d\theta\,d\phi \\ &= 2\pi\sqrt{\frac{9}{32\pi^2}}\int_0^{\pi}\sin^2\theta\cos\theta\,d\theta \\ &= \sqrt{\frac{9}{8}}\left(\frac{\sin^3\theta}{3}\right)\Bigg|_0^{\pi} = 0 \end{aligned}$$

$$\boxed{P_{2\hbar^2} = 0}$$

(b) $P_{6\hbar^2} = |c_{2,0}|^2 + |c_{2,+1}|^2 + |c_{2,-1}|^2 + |c_{2,+2}|^2 + |c_{2,-2}|^2$. Again, we know that only $c_{2,0}$ can be nonzero because there is no ϕ dependence in Ψ.

$$\begin{aligned} c_{2,0} &= \int_0^{2\pi}\int_0^{\pi}\sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)\cdot\sqrt{\frac{3}{8\pi}}\sin\theta\,\sin\theta\,d\theta\,d\phi \\ &= 2\pi\sqrt{\frac{15}{128\pi^2}}\int_0^{\pi}(3\sin^2\theta\cos^2\theta - \sin^2\theta)\,d\theta \\ &= \sqrt{\frac{15}{32}}\left(3\frac{\pi}{8} - \frac{\pi}{2}\right) = -\sqrt{\frac{15}{32}}\frac{\pi}{8} = -\sqrt{\frac{15\pi^2}{2048}} \end{aligned}$$

$$\boxed{P_{6\hbar^2} = \frac{15\pi^2}{2048} \approx 0.0723}$$

Question #3:

(a) $\hat{L}_z|\Psi(t)\rangle = \frac{1}{3}\left(-\hbar|1 \quad -1\rangle e^{+i\omega t} + 2\cdot 0\hbar|1 \quad 0\rangle + 2i\cdot\hbar|1 \quad +1\rangle e^{-i\omega t}\right)$

$\hat{L}_z|\Psi(t)\rangle = \frac{1}{3}\left(-\hbar|1 \quad -1\rangle e^{+i\omega t} + 2i\cdot\hbar|1 \quad +1\rangle e^{-i\omega t}\right)$

$\langle L_z\rangle =$

$$\frac{1}{3}\left(\langle 1 \quad -1|e^{-i\omega t} + 2\langle 1 \quad 0| - 2i\langle 1 \quad +1|e^{+i\omega t}\right)\frac{1}{3}\left(-\hbar|1 \quad -1\rangle e^{+i\omega t} + 2i\cdot\hbar|1 \quad +1\rangle e^{-i\omega t}\right)$$
$$= \frac{1}{9}\left(-\hbar + 4\hbar\right)$$

$$\boxed{\langle L_z(t)\rangle = \frac{\hbar}{3}}$$

$\hat{L}_+|\Psi(t)\rangle = \frac{1}{3}\left(\sqrt{2}\hbar|1 \quad 0\rangle e^{+i\omega t} + 2\sqrt{2}\hbar|1 \quad +1\rangle\right)$

$\hat{L}_-|\Psi(t)\rangle = \frac{1}{3}\left(2\sqrt{2}\hbar|1 \quad -1\rangle + \sqrt{2}\hbar 2i|1 \quad 0\rangle e^{-i\omega t}\right)$

(b) $\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$ so therefore

$\hat{L}_y|\Psi(t)\rangle = \frac{1}{6i}\left(-2\sqrt{2}\hbar|1 \quad -1\rangle + \sqrt{2}\hbar|1 \quad 0\rangle\left(e^{+i\omega t} - 2ie^{-i\omega t}\right) + 2\sqrt{2}\hbar|1 \quad +1\rangle\right)$

$$\langle L_y\rangle = \frac{1}{3}\left(\langle 1 \quad -1|e^{-i\omega t} + 2\langle 1 \quad 0| - 2i\langle 1 \quad +1|e^{+i\omega t}\right)$$
$$\frac{1}{6i}\left(-2\sqrt{2}\hbar|1 \quad -1\rangle + \sqrt{2}\hbar|1 \quad 0\rangle\left(e^{+i\omega t} - 2ie^{-i\omega t}\right) + 2\sqrt{2}\hbar|1 \quad +1\rangle\right)$$

$$\begin{aligned}
\langle L_y\rangle &= \frac{\hbar}{18i}\left(-2\sqrt{2}e^{-i\omega t} + 2\sqrt{2}\left(e^{+i\omega t} - 2ie^{-i\omega t}\right) - 4i\sqrt{2}e^{+i\omega t}\right) \\
&= \frac{\hbar}{18i}\left(2\sqrt{2}\left(e^{+i\omega t} - e^{-i\omega t}\right) - 4i\sqrt{2}\left(e^{+i\omega t} + e^{-i\omega t}\right)\right) \\
&= \frac{\hbar}{18i}\left(2\sqrt{2}\left(2i\sin\omega t\right) - 4i\sqrt{2}\left(2\cos\omega t\right)\right) \\
&= \frac{4\sqrt{2}\hbar}{18}\left(-2\cos\omega t + \sin\omega t\right)
\end{aligned}$$

$$\boxed{\langle L_y(t)\rangle = \frac{2\sqrt{2}\hbar}{9}\left(-2\cos\omega t + \sin\omega t\right)}$$