Question #1:

(a) $P_{z,+} = |0.6e^{+i\omega t}|^2 = 0.36$. 36% probability of measuring $+\hbar/2$.

$P_{z,-} = |0.8e^{-i\omega t}|^2 = 0.64$. 64% probability of measuring $-\hbar/2$.

(b) $P_{x,+} = |\langle \chi_+^{(x)} | \chi \rangle|^2 \qquad \chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\langle \chi_+^{(x)} | \chi \rangle = \tfrac{1}{\sqrt{2}}(1 \;\; 1)^* \begin{pmatrix} 0.6e^{+i\omega t} \\ 0.8e^{-i\omega t} \end{pmatrix} = \frac{1}{\sqrt{2}} \left(0.6e^{+i\omega t} + 0.8e^{-i\omega t}\right)$$

$$\begin{aligned} |\langle \chi_+^{(x)} | \chi \rangle|^2 &= \frac{1}{\sqrt{2}} \left(0.6e^{-i\omega t} + 0.8e^{+i\omega t}\right) \cdot \frac{1}{\sqrt{2}} \left(0.6e^{+i\omega t} + 0.8e^{-i\omega t}\right) \\ &= \frac{1}{2} \left(0.36 + 0.64 + 0.48e^{+2i\omega t} + 0.48e^{-2i\omega t}\right) \\ &= \frac{1}{2} \left(1 + 0.96\cos(2\omega t)\right) \end{aligned}$$

$$\boxed{P_{x,+} = (0.5 + 0.48\cos(2\omega t))}$$

$P_{x,-} = |\langle \chi_-^{(x)} | \chi \rangle|^2 \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\langle \chi_-^{(x)} | \chi \rangle = \tfrac{1}{\sqrt{2}}(1 \;\; -1)^* \begin{pmatrix} 0.6e^{+i\omega t} \\ 0.8e^{-i\omega t} \end{pmatrix} = \frac{1}{\sqrt{2}} \left(0.6e^{+i\omega t} - 0.8e^{-i\omega t}\right)$$

$$\begin{aligned} |\langle \chi_-^{(x)} | \chi \rangle|^2 &= \frac{1}{\sqrt{2}} \left(0.6e^{-i\omega t} - 0.8e^{+i\omega t}\right) \cdot \frac{1}{\sqrt{2}} \left(0.6e^{+i\omega t} - 0.8e^{-i\omega t}\right) \\ &= \frac{1}{2} \left(0.36 + 0.64 - 0.48e^{+2i\omega t} - 0.48e^{-2i\omega t}\right) \\ &= \frac{1}{2} \left(1 - 0.96\cos(2\omega t)\right) \end{aligned}$$

$$\boxed{P_{x,-} = (0.5 - 0.48\cos(2\omega t))}$$

Question #2:

(a) $\hat{H}\chi = -\frac{\gamma\hbar}{2}B_0\cos(\omega t)\frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ -e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix}$

$\frac{d}{dt}\chi = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \cdot (i\frac{\gamma B_0}{2\omega}\cos(\omega t)) \\ e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \cdot (-i\frac{\gamma B_0}{2\omega}\cos(\omega t)) \end{pmatrix}$

So $i\hbar\frac{d}{dt}\chi = -\frac{\gamma\hbar}{2}B_0\cos(\omega t)\frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ -e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix}$

(b)

$$\begin{aligned}
\langle S_x \rangle &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} & e^{+i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \\
&= \frac{\hbar}{4}\begin{pmatrix} e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} & e^{+i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ e^{+i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \\
&= \frac{\hbar}{4}\left(e^{-i\frac{\gamma B_0}{\omega}\sin(\omega t)} + e^{+i\frac{\gamma B_0}{\omega}\sin(\omega t)}\right)
\end{aligned}$$

$$\boxed{\langle S_x \rangle = \frac{\hbar}{2}\cos\left(\frac{\gamma B_0}{\omega}\sin(\omega t)\right)}$$

$$\begin{aligned}
\langle S_y \rangle &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} & e^{+i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \\
&= \frac{\hbar}{4}\begin{pmatrix} e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} & e^{+i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \begin{pmatrix} -ie^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ ie^{+i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \\
&= \frac{\hbar}{4}\left(-ie^{-i\frac{\gamma B_0}{\omega}\sin(\omega t)} + ie^{+i\frac{\gamma B_0}{\omega}\sin(\omega t)}\right) = i\frac{\hbar}{4}\left(e^{+i\frac{\gamma B_0}{\omega}\sin(\omega t)} - e^{-i\frac{\gamma B_0}{\omega}\sin(\omega t)}\right) \\
&= i\frac{\hbar}{4}\left(2i\sin\left(\frac{\gamma B_0}{\omega}\sin(\omega t)\right)\right)
\end{aligned}$$

$$\boxed{\langle S_y \rangle = -\frac{\hbar}{2}\sin\left(\frac{\gamma B_0}{\omega}\sin(\omega t)\right)}$$

$$\begin{aligned}
\langle S_z \rangle = & \tfrac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} & e^{+i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \\
&= \frac{\hbar}{4}\begin{pmatrix} e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} & e^{+i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \begin{pmatrix} e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ -e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \\
&= \frac{\hbar}{4}(1-1)
\end{aligned}$$

$$\boxed{\langle S_z \rangle = 0}$$

(c) $c_{x,-} = \langle \chi_-^{(x)} | \chi \rangle = \tfrac{1}{\sqrt{2}}(1\ \ -1)\ \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \\ e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} \end{pmatrix}$

$$c_{x,-} = \frac{1}{2}\left(e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} - e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)}\right) \quad \text{or} \quad c_{x,-} = i\sin\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)$$

$$\begin{aligned}
|c_{x,-}|^2 &= \frac{1}{2}\left(e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)} - e^{+i\frac{\gamma B_0}{2\omega}\sin(\omega t)}\right)\frac{1}{2}\left(e^{i\frac{\gamma B_0}{2\omega}\sin(\omega t)} - e^{-i\frac{\gamma B_0}{2\omega}\sin(\omega t)}\right) \\
&= \frac{1}{4}\left(1 - e^{-i\frac{\gamma B_0}{\omega}\sin(\omega t)} - e^{i\frac{\gamma B_0}{\omega}\sin(\omega t)} + 1\right) \\
&= \frac{1}{4}\left(2 - 2\cos\left(\frac{\gamma B_0}{\omega}\sin(\omega t)\right)\right)
\end{aligned}$$

$$\boxed{P_{x,-} = \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\gamma B_0}{\omega}\sin(\omega t)\right)\right) = \sin^2\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right)}$$

(d) If we want $P_{x,-} = 1$ then we need $\cos\left(\frac{\gamma B_0}{\omega}\sin(\omega t)\right) = -1$ so $\frac{\gamma B_0}{\omega}\sin(\omega t) = \pi$. The sine function is maximized at 1. So therefore

$\boxed{(B_0)_{min} = \frac{\pi\omega}{\gamma}}$ is the minimum field needed to force a complete spin flip.