

Question #1:

For the ground state ($\hbar\omega$) $n_1 + n_2 = 0$, so $n_1 = n_2 = 0$. For $E = 2\hbar\omega$, $n_1 + n_2 = 1$ which comes from $0 + 1 = 1$. For $E = 3\hbar\omega$, $n_1 + n_2 = 2$ which comes from either $0 + 2 = 2$ or $1 + 1 = 2$. For $E = 4\hbar\omega$, $n_1 + n_2 = 3$ which comes from either $0 + 3 = 3$ or $1 + 2 = 3$.

With distinguishable particles, each combination can go with any of the four spin states.

	Distinguishable Spin-1/2 Particles	
Energy	States	Degeneracy
$\hbar\omega$	$\psi_0\psi_0 0\ 0\rangle, \psi_0\psi_0 1\ m\rangle$	4
$2\hbar\omega$	$\psi_0\psi_1 0\ 0\rangle, \psi_0\psi_1 1\ m\rangle, \psi_1\psi_0 0\ 0\rangle, \psi_1\psi_0 1\ m\rangle$	8
$3\hbar\omega$	$\psi_1\psi_1 0\ 0\rangle, \psi_1\psi_1 1\ m\rangle, \psi_2\psi_0 0\ 0\rangle, \psi_2\psi_0 1\ m\rangle, \psi_0\psi_2 0\ 0\rangle, \psi_0\psi_2 1\ m\rangle$	12
$4\hbar\omega$	$\psi_0\psi_3 0\ 0\rangle, \psi_0\psi_3 1\ m\rangle, \psi_1\psi_2 0\ 0\rangle, \psi_1\psi_2 1\ m\rangle, \psi_2\psi_1 0\ 0\rangle, \psi_2\psi_1 1\ m\rangle, \psi_3\psi_0 0\ 0\rangle, \psi_3\psi_0 1\ m\rangle$	16

With identical fermions we must place spatially symmetric combinations with the singlet state or spatially antisymmetric combinations with the triplet state.

	Identical Spin-1/2 Fermions	
Energy	States	Degeneracy
$\hbar\omega$	$\psi_0\psi_0 0\ 0\rangle$	1
$2\hbar\omega$	$\frac{1}{\sqrt{2}}[\psi_0\psi_1 + \psi_1\psi_0] 0\ 0\rangle, \frac{1}{\sqrt{2}}[\psi_0\psi_1 - \psi_1\psi_0] 1\ m\rangle$	4
$3\hbar\omega$	$\psi_1\psi_1 0\ 0\rangle, \frac{1}{\sqrt{2}}[\psi_0\psi_2 + \psi_2\psi_0] 0\ 0\rangle, \frac{1}{\sqrt{2}}[\psi_0\psi_2 - \psi_2\psi_0] 1\ m\rangle$	5
$4\hbar\omega$	$\frac{1}{\sqrt{2}}[\psi_0\psi_3 + \psi_3\psi_0] 0\ 0\rangle, \frac{1}{\sqrt{2}}[\psi_0\psi_3 - \psi_3\psi_0] 1\ m\rangle, \frac{1}{\sqrt{2}}[\psi_1\psi_2 + \psi_2\psi_1] 0\ 0\rangle, \frac{1}{\sqrt{2}}[\psi_1\psi_2 - \psi_2\psi_1] 1\ m\rangle$	8

With distinguishable spin-1 particles, each combination can go with any of the now nine spin states.

	Distinguishable Spin-1 Particles	
Energy	States	Degeneracy
$\hbar\omega$	$\psi_0\psi_0 2\ m\rangle, \psi_0\psi_0 1\ m\rangle, \psi_0\psi_0 0\ 0\rangle$	9
$2\hbar\omega$	$\psi_0\psi_1 2\ m\rangle, \psi_0\psi_1 1\ m\rangle, \psi_0\psi_1 0\ 0\rangle,$ $\psi_1\psi_0 2\ m\rangle, \psi_1\psi_0 1\ m\rangle, \psi_1\psi_0 0\ 0\rangle$	18
$3\hbar\omega$	$\psi_0\psi_2 2\ m\rangle, \psi_0\psi_2 1\ m\rangle, \psi_0\psi_2 0\ 0\rangle,$ $\psi_1\psi_1 2\ m\rangle, \psi_1\psi_1 1\ m\rangle, \psi_1\psi_1 0\ 0\rangle,$ $\psi_2\psi_0 2\ m\rangle, \psi_2\psi_0 1\ m\rangle, \psi_2\psi_0 0\ 0\rangle$	27
$4\hbar\omega$	$\psi_0\psi_3 2\ m\rangle, \psi_0\psi_3 1\ m\rangle, \psi_0\psi_3 0\ 0\rangle,$ $\psi_1\psi_2 2\ m\rangle, \psi_1\psi_2 1\ m\rangle, \psi_1\psi_2 0\ 0\rangle,$ $\psi_2\psi_1 2\ m\rangle, \psi_2\psi_1 1\ m\rangle, \psi_2\psi_1 0\ 0\rangle,$ $\psi_3\psi_0 2\ m\rangle, \psi_3\psi_0 1\ m\rangle, \psi_3\psi_0 0\ 0\rangle$	36

For two spin-1 particles, the $|+2\ m\rangle$ quintuplet and $|0\ 0\rangle$ states are symmetric under spin exchange while the $|1\ m\rangle$ triplet states is antisymmetric. Note that for the singlet and triplet states this is opposite from what we say before for the two spin-1/2 particles. In any event, for identical bosons we can combine a spatially symmetric wave function with a spin symmetric function (singlet or quintuplet) or we can combine a spatially antisymmetric wave function with a spin antisymmetric triplet.

Identical Spin-1 Bosons		
Energy	States	Degeneracy
$\hbar\omega$	$\psi_0\psi_0 2\ m\rangle, \psi_0\psi_0 0\ 0\rangle$	6
$2\hbar\omega$	$\frac{1}{\sqrt{2}}[\psi_0\psi_1 - \psi_1\psi_0] 1\ m\rangle,$ $\frac{1}{\sqrt{2}}[\psi_0\psi_1 + \psi_1\psi_0] 0\ 0\rangle, \frac{1}{\sqrt{2}}[\psi_0\psi_1 + \psi_1\psi_0] 2\ m\rangle$	9
$3\hbar\omega$	$\psi_1\psi_1 0\ 0\rangle, \psi_1\psi_1 2\ m\rangle, \frac{1}{\sqrt{2}}[\psi_0\psi_2 - \psi_2\psi_0] 1\ m\rangle,$ $\frac{1}{\sqrt{2}}[\psi_0\psi_2 + \psi_2\psi_0] 0\ 0\rangle, \frac{1}{\sqrt{2}}[\psi_0\psi_2 + \psi_2\psi_0] 2\ m\rangle$	15
$4\hbar\omega$	$\frac{1}{\sqrt{2}}[\psi_0\psi_3 - \psi_3\psi_0] 1\ m\rangle,$ $\frac{1}{\sqrt{2}}[\psi_0\psi_3 + \psi_3\psi_0] 0\ 0\rangle, \frac{1}{\sqrt{2}}[\psi_0\psi_3 + \psi_3\psi_0] 2\ m\rangle$ $\frac{1}{\sqrt{2}}[\psi_1\psi_2 - \psi_2\psi_1] 1\ m\rangle,$ $\frac{1}{\sqrt{2}}[\psi_1\psi_2 + \psi_2\psi_1] 0\ 0\rangle, \frac{1}{\sqrt{2}}[\psi_1\psi_2 + \psi_2\psi_1] 2\ m\rangle$	18

Question 2:

(a)(i) Distinguishable particles:

$$\psi(x_1, x_2) = \psi_0(x_1)\psi_1(x_2) = \sqrt{\frac{2}{\pi}}e^{-x_1^2/2}x_2e^{-x_2^2/2}$$

$$|\psi(x_1, x_2)|^2 = \frac{2}{\pi}x_2^2e^{-(x_1^2+x_2^2)}$$

(ii) Identical bosons:

$$\begin{aligned}\psi(x_1, x_2) &= \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)] \\ &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{\pi}}e^{-x_1^2/2}x_2e^{-x_2^2/2} + \sqrt{\frac{2}{\pi}}x_1e^{-x_1^2/2}e^{-x_2^2/2}\right]\end{aligned}$$

$$\psi(x_1, x_2) = \frac{1}{\sqrt{\pi}}e^{-(x_1^2+x_2^2)/2}(x_1 + x_2)$$

$$|\psi(x_1, x_2)|^2 = \frac{1}{\pi}e^{-(x_1^2+x_2^2)}(x_1 + x_2)^2$$

(iii) Identical fermions:

$$\begin{aligned}\psi(x_1, x_2) &= \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)] \\ &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{\pi}}e^{-x_1^2/2}x_2e^{-x_2^2/2} - \sqrt{\frac{2}{\pi}}x_1e^{-x_1^2/2}e^{-x_2^2/2}\right]\end{aligned}$$

$$\psi(x_1, x_2) = \frac{1}{\sqrt{\pi}}e^{-(x_1^2+x_2^2)/2}(x_2 - x_1)$$

$$|\psi(x_1, x_2)|^2 = \frac{1}{\pi}e^{-(x_1^2+x_2^2)}(x_2 - x_1)^2$$

(b)(i)

$$\begin{aligned}P(|r|) &= 2 \int_{-\infty}^{\infty} \frac{2}{\pi} \left(R - \frac{r}{2}\right)^2 e^{-(\frac{1}{2}r^2 + 2R^2)} dR \\ &= \frac{4}{\pi}e^{-r^2/2} \int_{-\infty}^{\infty} \left(R^2 - Rr + \frac{r^2}{4}\right) e^{-2R^2} dR = \frac{4}{\pi}e^{-r^2/2} \left[\frac{1}{2}\sqrt{\frac{\pi}{8}} - r \cdot 0 + \frac{r^2}{4} \cdot \sqrt{\frac{\pi}{2}}\right]\end{aligned}$$

$$P(|r|) = \frac{1}{\sqrt{2\pi}}e^{-r^2/2}(1 + r^2) \text{ for distinguishable particles}$$

(ii)

$$P(|r|) = 2 \int_{-\infty}^{\infty} \frac{1}{\pi} (4R^2) e^{-(\frac{1}{2}r^2 + 2R^2)} dR$$

$$= \frac{8}{\pi} e^{-r^2/2} \int_{-\infty}^{\infty} (R^2) e^{-2R^2} dR = \frac{8}{\pi} e^{-r^2/2} \left[\frac{1}{2} \sqrt{\frac{\pi}{8}} \right]$$

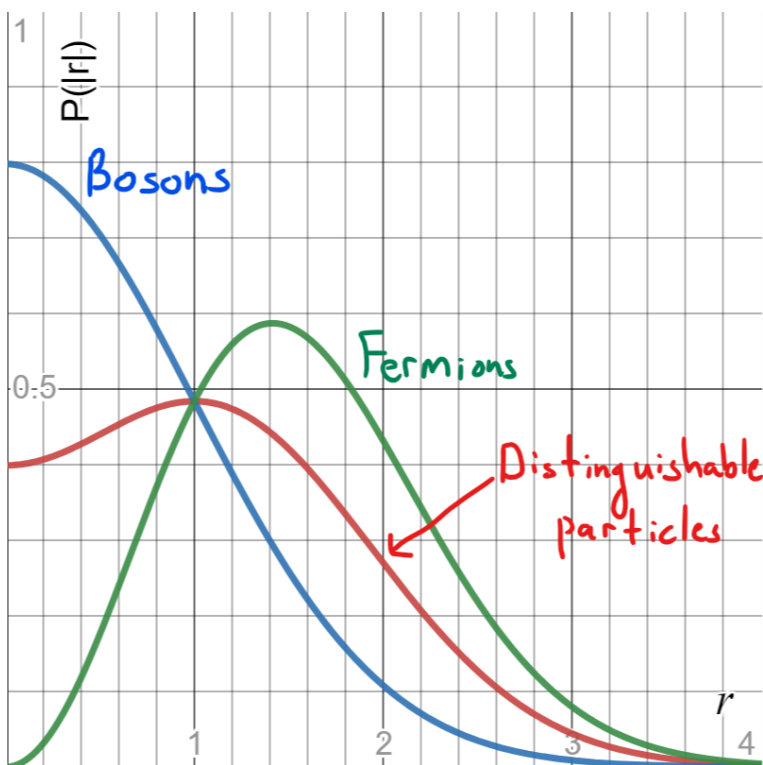
$$P(|r|) = \sqrt{\frac{2}{\pi}} e^{-r^2/2} \text{ for identical bosons}$$

(iii)

$$P(|r|) = 2 \int_{-\infty}^{\infty} \frac{1}{\pi} (r^2) e^{-(\frac{1}{2}r^2 + 2R^2)} dR$$

$$= \frac{2}{\pi} r^2 e^{-r^2/2} \int_{-\infty}^{\infty} e^{-2R^2} dR = \frac{2}{\pi} r^2 e^{-r^2/2} \left[\sqrt{\frac{\pi}{2}} \right]$$

$$P(|r|) = \sqrt{\frac{2}{\pi}} r^2 e^{-r^2/2} \text{ for identical fermions}$$



Question #3: (a) From NIST, listed in increasing order of energy:

$1s^2$	0 cm^{-1}	Parahelium
$1s2s$	159860 cm^{-1}	Orthohelium
$1s2s$	166280 cm^{-1}	Parahelium
$1s2p$	169090 cm^{-1}	Orthohelium
$1s2p$	171130 cm^{-1}	Parahelium
$1s3s$	183240 cm^{-1}	Orthohelium
$1s3s$	184860 cm^{-1}	Parahelium
$1s3p$	185560 cm^{-1}	Orthohelium
$1s3d$	186100 cm^{-1}	Orthohelium
$1s3d$	186100 cm^{-1}	Parahelium
$1s3p$	186210 cm^{-1}	Parahelium

(b) For the parahelium $1s2s$ state:

$$(166280 \text{ cm}^{-1} - 637220 \text{ cm}^{-1}) \cdot \frac{1 \text{ eV}}{8065.8 \text{ cm}^{-1}} = -58.387 \text{ eV}$$

$$\frac{e^2}{4\pi\epsilon_0} = 2.307 \times 10^{-28} \text{ J} \cdot \text{m} = 1.440 \text{ eV} \cdot \text{nm}$$

$$\frac{e^2}{4\pi\epsilon_0} \frac{1}{\langle |\vec{r}_2 - \vec{r}_1| \rangle} = +9.643 \text{ eV} \text{ so therefore } \boxed{\langle |\vec{r}_2 - \vec{r}_1| \rangle = 0.149 \text{ nm}}$$

For the orthohelium $1s2s$ state:

$$(159860 \text{ cm}^{-1} - 637220 \text{ cm}^{-1}) \cdot \frac{1 \text{ eV}}{8065.8 \text{ cm}^{-1}} = -59.183 \text{ eV}$$

$$\frac{e^2}{4\pi\epsilon_0} \frac{1}{\langle |\vec{r}_2 - \vec{r}_1| \rangle} = +8.847 \text{ eV} \text{ so therefore } \boxed{\langle |\vec{r}_2 - \vec{r}_1| \rangle = 0.163 \text{ nm}}$$

Parahelium states are in a spatially symmetric wavefunction, so they are, on average, a little closer together.

Helium Energy Level Diagram (ignoring fine structure)

