Question #1:(a)

$$E_1^{(1)} = \langle\psi_1^{(0)}|\hat{H}_1|\psi_1^{(0)}\rangle = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\,\alpha\delta\left(x-\frac{a}{2}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\,dx$$
$$= \alpha\frac{2}{a}\sin^2\left(\frac{\pi}{2}\right) = \frac{2\alpha}{a}$$

(b) $|\psi_1^{(1)}\rangle = \sum_{m\neq 1}\frac{\langle\psi_m^{(0)}|\hat{H}_1|\psi_1^{(0)}\rangle}{E_1^{(0)}-E_m^{(0)}}|\psi_m^{(0)}\rangle$

$$\langle\psi_m^{(0)}|\hat{H}_1|\psi_1^{(0)}\rangle = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right)\,\alpha\delta\left(x-\frac{a}{2}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\,dx = \frac{2\alpha}{a}\sin\left(\frac{m\pi}{2}\right)$$

Therefore:

$$\langle\psi_m^{(0)}|\hat{H}_1|\psi_1^{(0)}\rangle = \begin{cases} 0 & m \text{ is even} \\ +\frac{2\alpha}{a} & m = 4n+1 \text{ for integer } n \\ -\frac{2\alpha}{a} & m = 4n+3 \text{ for integer } n \end{cases}$$

$E_m^{(0)} = m^2 E_1^{(0)}$ so the denominator $E_1^{(0)} - E_m^{(0)}$ is equal to $-(m^2-1)E_1^{(0)}$.

The first three nonzero terms will be when m is 3, 5, or 7.

$$\psi_1^{(1)} \approx \left(-\frac{2\alpha}{a}\right)\frac{1}{-8E_1^{(0)}}\sqrt{\frac{2}{a}}\sin\left(\frac{3\pi}{a}x\right)$$
$$+\left(+\frac{2\alpha}{a}\right)\frac{1}{-24E_1^{(0)}}\sqrt{\frac{2}{a}}\sin\left(\frac{5\pi}{a}x\right) + \left(-\frac{2\alpha}{a}\right)\frac{1}{-48E_1^{(0)}}\sqrt{\frac{2}{a}}\sin\left(\frac{7\pi}{a}x\right)$$
$$\approx \sqrt{\frac{2}{a}}\frac{\alpha}{aE_1^{(0)}}\left[\frac{1}{4}\sin\left(\frac{3\pi}{a}x\right) - \frac{1}{12}\sin\left(\frac{5\pi}{a}x\right) + \frac{1}{24}\sin\left(\frac{7\pi}{a}x\right)\right]$$

(c) $E_1^{(2)} = \sum_{m\neq 1}\frac{|\langle\psi_m^{(0)}|\hat{H}_1|\psi_1^{(0)}\rangle|^2}{E_1^{(0)}-E_m^{(0)}}$. This uses many of the same terms from the previous part. Again, the first three nonzero terms come from $m = 3$, 5, and 7.

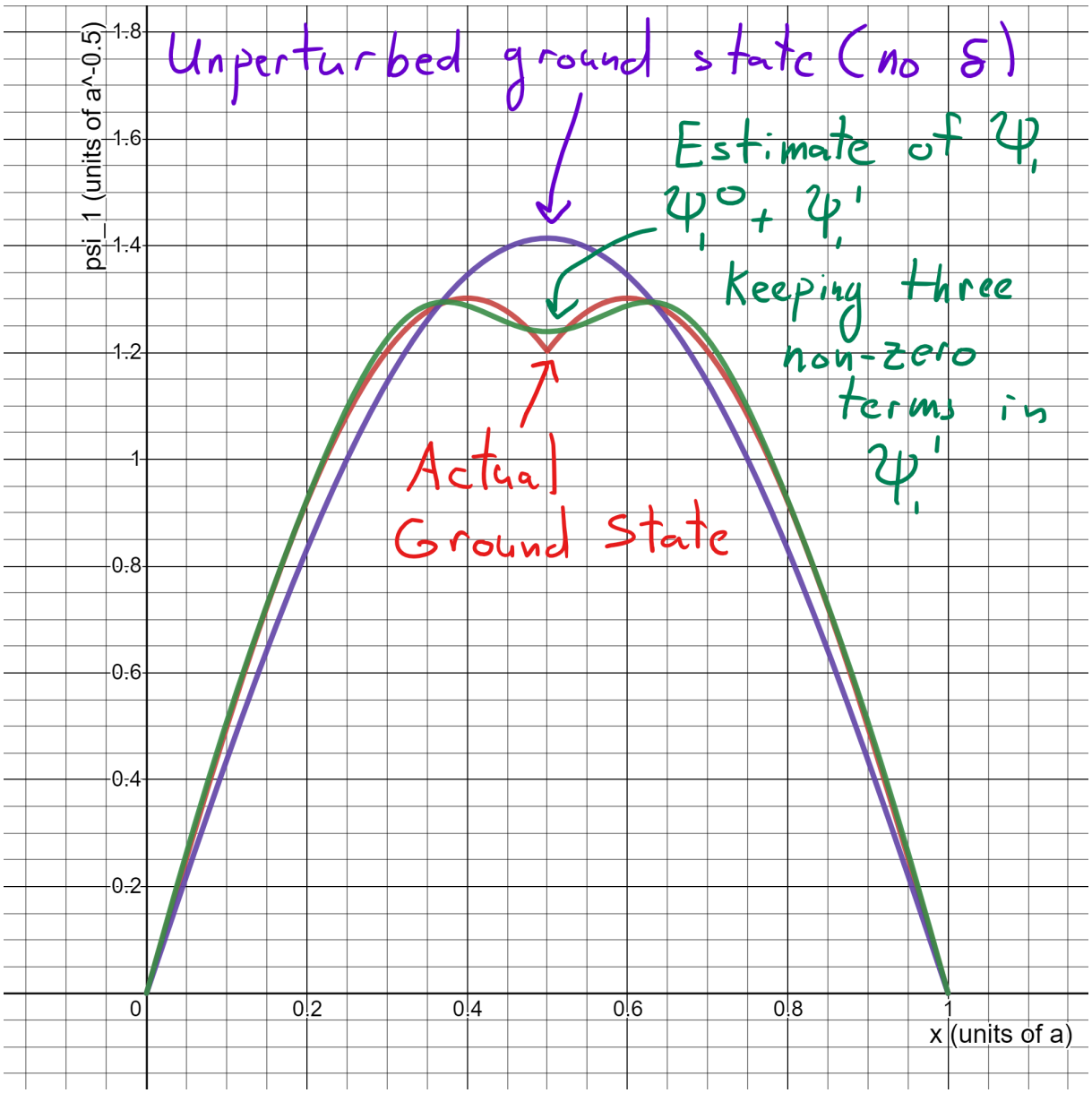
$$E_1^{(2)} \approx \frac{1}{-8E_1^{(0)}}\left(\frac{2\alpha}{a}\right)^2 + \frac{1}{-24E_1^{(0)}}\left(\frac{2\alpha}{a}\right)^2 + \frac{1}{-48E_1^{(0)}}\left(\frac{2\alpha}{a}\right)^2 \approx -\frac{3\alpha^2}{4E_1^{(0)}a^2}$$

(d) $E_1 \approx E_1^{(0)} + E_1^{(1)} + E_1^{(2)} \approx E_1^{(0)} + \frac{2\alpha}{a} - \frac{3\alpha^2}{4E_1^{(0)}a^2}$. Setting $E_1^{(0)} = \frac{\hbar^2\pi^2}{2ma^2}$ and $\alpha = 1.63\frac{\hbar^2}{ma}$ gives $\boxed{E_1 \approx 1.58E_1^{(0)} \approx 7.79\frac{\hbar^2}{ma^2}}$ This is about 1% larger than the real energy.

Keeping the first three non-zero terms in the first order correction our estimate of the ground state wave function is

$$\psi_1(x) \approx \sqrt{\frac{2}{a}}\left[\sin\left(\frac{\pi x}{a}\right) + 0.330\left[\frac{1}{4}\sin\left(\frac{3\pi}{a}x\right) - \frac{1}{12}\sin\left(\frac{5\pi}{a}x\right) + \frac{1}{24}\sin\left(\frac{7\pi}{a}x\right)\right]\right]$$

This is shown below in green.



Question #1:

(a) $E_0^1 = \langle \psi_0^0 | \hat{H}' | \psi_0^0 \rangle$

$$E_0^1 = \int_{-L/2}^{L/2} \left(\frac{1}{\sqrt{L}}\right)^* (V_0 L \delta(x)) \left(\frac{1}{\sqrt{L}}\right) dx = V_0$$

(b) $W_{aa} = \langle \psi_a^0 | \hat{H}' | \psi_a^0 \rangle$.

$$W_{aa} = \int_{-L/2}^{L/2} \left(\frac{1}{\sqrt{L}} e^{i2\pi nx/L}\right)^* (V_0 L \delta(x)) \left(\frac{1}{\sqrt{L}} e^{i2\pi nx/L}\right) dx$$
$$= V_0 \int_{-L/2}^{L/2} \delta(x)\, dx = V_0$$

$$W_{ab} = \int_{-L/2}^{L/2} \left(\frac{1}{\sqrt{L}} e^{i2\pi nx/L}\right)^* (V_0 L \delta(x)) \left(\frac{1}{\sqrt{L}} e^{-i2\pi nx/L}\right) dx$$
$$= \frac{V_0}{L} \int_{-L/2}^{L/2} e^{-i4\pi nx/L}\, \delta(x)\, dx = V_0$$

$W_{bb} = W_{aa}$ and $W_{ba} = (W_{ab})^* = W_{ab}$

(c) $\boxed{E_\pm^1 = \frac{1}{2}\left[2V_0 \pm 2V_0\right] = V_0\left[1 \pm 1\right]}$, $E_+^1 = 2V_0$, $E_-^1 = 0$

(d)

$$\begin{pmatrix} V_0 & V_0 \\ V_0 & V_0 \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix} = 2V_0 \begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix}$$

$V_0\alpha_+ + V_0\beta_+ = 2V_0\alpha_+$ so $\beta_+ = \alpha_+$

$$\boxed{\psi_+ = \frac{1}{\sqrt{2}}\left(\psi_a^0 + \psi_b^0\right) = \frac{1}{\sqrt{2L}}\left(e^{+i2\pi nx/L} + e^{-i2\pi nx/L}\right) = \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi n}{L}x\right)}$$

$$\begin{pmatrix} V_0 & V_0 \\ V_0 & V_0 \end{pmatrix} \begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix} = 0 \begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix}$$

$V_0\alpha_- + V_0\beta_- = 0$ so $\beta_- = -\alpha_-$

$$\boxed{\psi_- = \frac{1}{\sqrt{2}}\left(\psi_a^0 - \psi_b^0\right) = \frac{1}{\sqrt{2L}}\left(e^{+i2\pi nx/L} - e^{-i2\pi nx/L}\right) = \sqrt{\frac{2}{L}}\, i \sin\left(\frac{2\pi n}{L}x\right)}$$

(e) $\hat{\Pi}\psi_+ = \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi n}{L}(-x)\right) = \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi n}{L}x\right) = +1 \cdot \psi_+$ Eigenvalue $+1$

$\hat{\Pi}\psi_- = \sqrt{\frac{2}{L}} i \sin\left(\frac{2\pi n}{L}(-x)\right) = -\sqrt{\frac{2}{L}} i \sin\left(\frac{2\pi n}{L}x\right) = -1 \cdot \psi_-$ Eigenvalue -1

Question #3:(a)

$$\begin{vmatrix} (1-\lambda) & \delta\, e^{+i\phi} \\ \delta\, e^{-i\phi} & (2-\lambda) \end{vmatrix} = (1-\lambda)(2-\lambda) - |\delta|^2 = \lambda^2 - 3\lambda + 2 - |\delta|^2 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9 - 4(2-\delta^2)}}{2} \quad \text{so} \quad \lambda = \frac{3}{2} \pm \frac{1}{2}\sqrt{1+4\delta^2}$$

$$\boxed{E_1 = E_0\left(\frac{3}{2} - \frac{1}{2}\sqrt{1+4\delta^2}\right)} \quad \text{and} \quad \boxed{E_2 = E_0\left(\frac{3}{2} + \frac{1}{2}\sqrt{1+4\delta^2}\right)}$$

Binomial expansion: $(1+4\delta^2)^{1/2} \approx 1 + \frac{1}{2}\cdot 4\delta^2$

$\lambda \approx \frac{3}{2} \pm \frac{1}{2}\left(1+2\delta^2\right)$ so $\boxed{E_1 \approx E_0(1-\delta^2)}$ and $\boxed{E_2 \approx E_0(2+\delta^2)}$

(b)

$$E_1^1 = \langle\psi_1^0|\hat{H}'|\psi_1^0\rangle = \begin{pmatrix}1 & 0\end{pmatrix}^* E_0 \begin{pmatrix} 0 & \delta e^{+i\phi} \\ \delta e^{-i\phi} & 0 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = E_0 \begin{pmatrix}1 & 0\end{pmatrix}^* \begin{pmatrix}0\\ \delta e^{-i\phi}\end{pmatrix} = 0$$

$$E_2^1 = \langle\psi_2^0|\hat{H}'|\psi_2^0\rangle = \begin{pmatrix}0 & 1\end{pmatrix}^* E_0 \begin{pmatrix} 0 & \delta e^{+i\phi} \\ \delta e^{-i\phi} & 0 \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = E_0 \begin{pmatrix}0 & 1\end{pmatrix}^* \begin{pmatrix}\delta e^{+i\phi}\\ 0\end{pmatrix} = 0$$

$$\langle\psi_1^0|\hat{H}'|\psi_2^0\rangle = \begin{pmatrix}1 & 0\end{pmatrix}^* E_0 \begin{pmatrix} 0 & \delta e^{+i\phi} \\ \delta e^{-i\phi} & 0 \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = E_0 \begin{pmatrix}1 & 0\end{pmatrix}^* \begin{pmatrix}\delta e^{+i\phi}\\ 0\end{pmatrix} = \delta e^{+i\phi} E_0$$

$\langle\psi_2^0|\hat{H}'|\psi_1^0\rangle = \langle\psi_1^0|\hat{H}'|\psi_2^0\rangle^* = \delta e^{-i\phi} E_0$

$$E_1^2 = \frac{|\delta^{-i\phi} E_0|^2}{E_0 - 2E_0} = -\delta^2 E_0 \qquad E_2^2 = \frac{|\delta e^{+i\phi} E_0|^2}{2E_0 - E_0} = \delta^2 E_0$$

$$\boxed{E_1 \approx E_0 + 0 - \delta^2 E_0 \approx E_0(1-\delta^2)} \qquad \boxed{E_2 \approx 2E_0 + 0 + \delta^2 E_0 \approx E_0(2+\delta^2)}$$

(c)

$$|\psi_1^1\rangle = \frac{\langle\psi_2^0|\hat{H}'|\psi_1^0\rangle}{E_0 - 2E_0}|\psi_2^0\rangle = \frac{\delta\, e^{-i\phi}E_0}{-E_0}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-\delta\, e^{-i\phi}\end{pmatrix}$$

$$\boxed{|\psi_1\rangle \approx |\psi_1^0\rangle + |\psi_1^1\rangle = \begin{pmatrix}1\\-\delta\, e^{-i\phi}\end{pmatrix}}$$

$$|\psi_2^1\rangle = \frac{\langle\psi_1^0|\hat{H}'|\psi_2^0\rangle}{2E_0 - E_0}|\psi_1^0\rangle = \frac{\delta\, e^{+i\phi}E_0}{E_0}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\delta\, e^{+i\phi}\\0\end{pmatrix}$$

$$\boxed{|\psi_2\rangle \approx |\psi_2^0\rangle + |\psi_2^1\rangle = \begin{pmatrix}\delta\, e^{+i\phi}\\1\end{pmatrix}}$$