Question #1: (a)

$$\langle T\rangle = -\frac{\hbar^2}{2m}\int \psi^* \frac{d^2\psi}{dx^2}\,dx = -\frac{\hbar^2}{2m}\left(8\sqrt{\frac{2b^3}{\pi}}\right)\int_0^\infty xe^{-bx^2}\,(4b^2x^3-6bx)e^{-bx^2}\,dx$$

$$= -\frac{\hbar^2}{2m}\left(8\sqrt{\frac{2b^3}{\pi}}\right)\int_0^\infty (4b^2x^4-6bx^2)e^{-2bx^2}\,dx$$

$$= -\frac{\hbar^2}{2m}\left(8\sqrt{\frac{2b^3}{\pi}}\right)\left[4b^2\cdot\sqrt{\frac{\pi}{2}}\frac{3}{32\,b^{5/2}} - 6b\cdot\sqrt{\frac{\pi}{2}}\frac{1}{8\,b^{3/2}}\right]$$

$$= -\frac{\hbar^2}{2m}\left(8\sqrt{\frac{2}{\pi}}\right)b^{3/2}\left[\frac{3b^{-1/2}}{8}\sqrt{\frac{\pi}{2}} - \frac{3b^{-1/2}}{4}\sqrt{\frac{\pi}{2}}\right] = -\frac{\hbar^2}{2m}\left(8\sqrt{\frac{2}{\pi}}\right)b^{3/2}\left[-\frac{3b^{-1/2}}{8}\sqrt{\frac{\pi}{2}}\right]$$

$$= \frac{3\hbar^2 b}{2m}$$

So $\boxed{\langle T\rangle = \frac{3\hbar^2 b}{2m}}$

(b)

$$\langle V\rangle = \left(8\sqrt{\frac{2b^3}{\pi}}\right)\int_0^\infty xe^{-bx^2}\,\alpha x\,xe^{-bx^2}\,dx = \alpha\left(8\sqrt{\frac{2b^3}{\pi}}\right)\int_0^\infty x^3\,e^{-2bx^2}\,dx$$

$$= \alpha\left(8\sqrt{\frac{2b^3}{\pi}}\right)\cdot\frac{1}{8b^2} = \sqrt{\frac{2}{\pi}}\alpha\,b^{-1/2}$$

So $\boxed{\langle V\rangle = \sqrt{\frac{2}{\pi}}\frac{\alpha}{\sqrt{b}}}$

(c) $\langle H\rangle = \frac{3\hbar^2 b}{2m} + \sqrt{\frac{2}{\pi}}\frac{\alpha}{\sqrt{b}}$

$\frac{d}{db}\langle H\rangle = \frac{3\hbar^2}{2m} - \frac{1}{2}\sqrt{\frac{2}{\pi}}\alpha b^{-3/2} = 0$ Therefore $b^{3/2} = \sqrt{\frac{2}{\pi}}\alpha\cdot\frac{m}{3\hbar^2}$

$$\boxed{b = \left(\sqrt{\frac{2}{\pi}}\alpha\cdot\frac{m}{3\hbar^2}\right)^{2/3}}$$

(d) $\langle H\rangle = \dfrac{3\hbar^2}{2m}\left(\sqrt{\dfrac{2}{\pi}}\alpha\cdot\dfrac{m}{3\hbar^2}\right)^{2/3} + \alpha\sqrt{\dfrac{2}{\pi}}\left(\sqrt{\dfrac{2}{\pi}}\alpha\cdot\dfrac{m}{3\hbar^2}\right)^{-1/3}$

Simplifying

$$\boxed{\langle H\rangle = \left[\frac{3}{2}\left(\frac{1}{3}\sqrt{\frac{2}{\pi}}\right)^{2/3} + \sqrt{\frac{2}{\pi}}\left(\frac{1}{3}\sqrt{\frac{2}{\pi}}\right)^{-1/3}\right]\left(\frac{\hbar^2\alpha^2}{m}\right)^{1/3} \approx 1.8611\left(\frac{\hbar^2\alpha^2}{m}\right)^{1/3}}$$

Only 0.3% larger than the actual ground state energy!

Question #2:

(a) Normalize: $4\pi \int_0^\infty A^2 e^{-2cr} r^2 \, dr = 4\pi A^2 \frac{2}{(2c)^3} = \frac{\pi A^2}{c^3} = 1$

$$\boxed{A = \sqrt{\frac{c^3}{\pi}}}$$

(b) We'll start with $\langle T \rangle$. $\frac{d}{dr}\psi = -c\sqrt{\frac{c^3}{\pi}} e^{-cr}$.

Therefore $\frac{d}{dr}\left(r^2 \frac{d}{dr}\psi\right) = -c\sqrt{\frac{c^3}{\pi}} \frac{d}{dr}\left(r^2 e^{-cr}\right) = -c\sqrt{\frac{c^3}{\pi}}(2r - cr^2)e^{-cr}$.

This gives $\hat{T}\psi = \frac{\hbar^2}{2m} c\sqrt{\frac{c^3}{\pi}} \left(\frac{2}{r} - c\right) e^{-cr}$

$$\begin{aligned}
\langle T \rangle &= 4\pi \int_0^\infty \left[\sqrt{\frac{c^3}{\pi}} e^{-cr}\right]^* \frac{\hbar^2}{2m} c\sqrt{\frac{c^3}{\pi}} \left(\frac{2}{r} - c\right) e^{-cr} \, r^2 \, dr \\
&= \frac{2c^4\hbar^2}{m} \int_0^\infty (2r - cr^2) e^{-2cr} \, dr \\
&= \frac{2c^4\hbar^2}{m} \left(2\frac{1}{(2c)^2} - c\frac{2}{(2c)^3}\right) \\
&= \frac{2c^4\hbar^2}{m} \left(\frac{1}{4c^2}\right) = \frac{\hbar^2 c^2}{2m}
\end{aligned}$$

Therefore $\boxed{\langle T \rangle = \frac{\hbar^2 c^2}{2m}}$

Now we do $\langle V \rangle$.

$$\begin{aligned}
\langle V \rangle &= \frac{-Ze^2}{4\pi\epsilon_0} 4\pi \int_0^\infty \left[\sqrt{\frac{c^3}{\pi}} e^{-cr}\right]^* \frac{1}{r} \left[\sqrt{\frac{c^3}{\pi}} e^{-cr}\right] r^2 \, dr \\
&= \frac{-Ze^2}{4\pi\epsilon_0} 4c^3 \int_0^\infty r e^{-2cr} \, dr \\
&= \frac{-Ze^2}{4\pi\epsilon_0} 4c^3 \frac{1}{(2c)^2}
\end{aligned}$$

Therefore $\boxed{\langle V \rangle = \frac{-Ze^2}{4\pi\epsilon_0} c}$

(c) $\langle H\rangle = \dfrac{\hbar^2 c^2}{2m} - \dfrac{Ze^2}{4\pi\epsilon_0}c \qquad \dfrac{d\langle H\rangle}{dc} = \dfrac{\hbar^2 c}{m} - \dfrac{Ze^2}{4\pi\epsilon_0} = 0$

This gives $\boxed{c = \dfrac{Zme^2}{4\pi\epsilon_0\hbar^2} = Z/a}$

(d) $\boxed{\langle T\rangle = \dfrac{Z^2\hbar^2}{2ma^2}}$

$\langle V\rangle = -Z^2\dfrac{e^2}{4\pi\epsilon_0}\dfrac{1}{a}$. Note that $\dfrac{e^2}{4\pi\epsilon_0} = \dfrac{\hbar^2}{ma}$. This gives $\boxed{\langle V\rangle = \dfrac{-Z^2\hbar^2}{ma^2}}$.

Clearly $\boxed{\langle H\rangle = \dfrac{-Z^2\hbar^2}{2ma^2}}$.

Interesting notes: $\dfrac{\hbar^2}{2ma^2} = 13.6$ eV. We also see that $\langle V\rangle = -2\langle T\rangle$, or $\langle T\rangle = -\dfrac{1}{2}\langle V\rangle$. This is a consequence of the Virial theorem: when $V \propto r^n$ we will find that $\langle T\rangle = \dfrac{n}{2}\langle V\rangle$. Again, these work because we chose a trial wavefunction that gives us the actual ground state wavefunction and energy after we apply the variational principle.