Question #1: (a) $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. Prove that $[\hat{L}^2,\, \hat{L}_z] = 0$.

(b) Prove that $[\hat{L}_\pm,\, \hat{L}_z] = \mp\hbar\hat{L}_\pm$, where $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$.

You may use the fact that $[\hat{L}_x,\, \hat{L}_y] = i\hbar\hat{L}_z$ (plus its cyclic permutations) as well as the fact that $[\hat{A}^2,\, \hat{B}] = \hat{A}[\hat{A},\, \hat{B}] + [\hat{A},\, \hat{B}]\hat{A}$.

Question #2: The wave function of a particle constrained to the surface of a sphere is given by

$$\Psi(\theta, \phi) = \sqrt{\frac{3}{8\pi}}\sin\theta$$

You make a measurement of L^2.

(a) What is the probability of measuring 0?

(b) What is the probability of measuring $2\hbar^2$?

(c) What is the probability of measuring $6\hbar^2$?

Note: the relevant spherical harmonics are given in Table 4.3 on page 137 of Griffiths & Schroeter. This wave function has no ϕ dependence so it is clear that any measurement of $\hat{L}_z$ will return $0\hbar$ as the z-component of the angular momentum.

$$d\Omega = \sin\theta\, d\theta\, d\phi, \qquad \int_0^\pi \sin^2\theta\, d\theta = \frac{\pi}{2}, \qquad \text{and} \qquad \int_0^\pi \sin^2\theta\cos^2\theta\, d\theta = \frac{\pi}{8}$$

Question #3: Suppose that the ket state of a spin-1 particle is given by

$$|\Psi(t)\rangle = \frac{1}{3}\left(|1 \quad -1\rangle e^{+i\omega t} + 2|1 \quad 0\rangle + 2i|1 \quad +1\rangle e^{-i\omega t}\right)$$

(a) What is $\langle L_z\rangle$?

(b) What is $\langle L_y\rangle$?

In part (b) do so by writing $\hat{L}_y$ in terms of $\hat{L}_+$ and $\hat{L}_-$ and finding $\langle\Psi|\hat{L}_y|\Psi\rangle$, using $\hat{L}_\pm|l \;\; m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|l \;\; m\pm 1\rangle$. Reminder: the expectation value of a physical observable is necessarily real, so your answers should not contain any i in it. Using Euler's relation you have $(e^{+i\omega t}+e^{-i\omega t}) = 2\cos\omega t$ and $(e^{+i\omega t}-e^{-i\omega t}) = 2i\sin\omega t$