

Question #1: Suppose we have the potential

$$V(x) = \begin{cases} \alpha x & x \geq 0 \\ \infty & x < 0 \end{cases}$$

We will apply the variational principle using the normalized test function

$$\psi(x) = \left( 8\sqrt{\frac{2b^3}{\pi}} \right)^{1/2} x e^{-bx^2}$$

- (a) What is the expectation value of the kinetic energy  $\langle T \rangle$ ?
- (b) What is the expectation value of the potential energy  $\langle V \rangle$ ?
- (c) What value of  $b$  minimizes the total energy?
- (d) What is your bound on the ground state energy? The actual ground state energy is  $E_{\text{g.s.}} \approx 1.8558 \left( \frac{\alpha^2 \hbar^2}{m} \right)^{1/3}$ . How much larger (as a ratio) is your bound? (This is a good test function - It will be quite close.)

Question #2: In this problem use  $\psi(\vec{r}) = Ae^{-cr}$  as your three-dimensional trial wave function. We are going to apply this function to a hydrogen-like potential (one electron orbiting a nucleus with  $Z$  protons.)

(a) Determine the normalization constant  $A$  in terms of  $c$ .

(b) Determine  $\langle T \rangle$  and  $\langle V \rangle$ . (Note: for a spherically symmetric wave function the kinetic energy operator reduces to  $\hat{T} = \frac{-\hbar^2}{2m} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right)$ . The potential energy is just Coulomb attraction:  $\hat{V} = \frac{-Ze^2}{4\pi\epsilon_0} \frac{1}{r}$ .)

(c) Determine the value of  $c$  that minimizes the energy. Then express this value of  $c$  as a function of  $Z$  and the Bohr radius ( $a \equiv \frac{4\pi\epsilon_0\hbar^2}{me^2}$ ).

(d) Write  $\langle T \rangle$ ,  $\langle V \rangle$ , and  $\langle H \rangle$  all with the value of  $c$  from the previous part as functions of only  $Z$ ,  $a$ ,  $\hbar$ , and  $m$ . (It turns out that this will be the actual ground state.)

Hints: Integrating a spherically symmetric function  $f(r)$  over all space in spherical coordinates is just  $4\pi \int_0^\infty f(r) r^2 dr$ . Use the integrals below or use Wolfram Alpha, integral-calculator.com, or similar.

$$\int_0^\infty r e^{-br} dr = \frac{1}{b^2} \quad \text{and} \quad \int_0^\infty r^2 e^{-br} dr = \frac{2}{b^3} \quad \text{assuming that } b > 0.$$