The Fundamental Postulates of Quantum Mechanics

- The state of a quantum particle or system can be completely represented by the quantity $|\Psi\rangle$. We can call this a ket state.
- Ket states evolve in time according to the Schrödinger equation: $i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$, where $\hat{H}$ is the Hamiltonian or total energy operator.
- There is a mathematical operation known as the inner product. $\langle\phi|\psi\rangle$ represents the inner product of state $|\phi\rangle$ with state $|\psi\rangle$.
  - Ket states are normalized: $\langle\Psi|\Psi\rangle = 1$.
- Physical observables are represented by hermitian operators. These operators will have a set of eigenstates, and their eigenvalues will always be real: $\hat{Q}|f_n\rangle = q_n|f_n\rangle$ where all q_n are real.
  - It is always possible to construct a set of eigenstates that are mutually orthonormal, and we will always choose to do so: $\langle f_m|f_n\rangle = \delta_{m,n}$. The eigenstates are normalized, and different eigenstates are orthogonal to each other.
  - The set of eigenstates of a hermitian operator will be complete. Any ket state can be written as a linear superposition of the eigenstates of the operator. Thus any $|\Psi\rangle$ can be written as

    $$|\Psi\rangle = \sum_n c_n|f_n\rangle$$

    where the expansion coefficients c_n can be determined from the inner product of their eigenstate with the particle's ket state: $c_n = \langle f_n|\Psi\rangle$.
  - Suppose that $|\Psi(t=0)\rangle = \sum_n c_n|\psi_n\rangle$ where the $|\psi_n\rangle$ are the energy eigenstates: $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$. The time-dependent ket

state is therefore given by

$$|\Psi(t)\rangle = \sum_n c_n |\psi_n\rangle e^{-iE_n t/\hbar}$$

- If you make a measurement of any physical observable, the observed result will always be one of the eigenvalues of the hermitian operator associated with this observable (one of the q_n). The probability of measuring any particular eigenvalue will be the norm-square of the inner product of the eigenstate that has that eigenvalue with the ket state of the particle: $P_n = |c_n|^2 = |\langle f_n|\Psi\rangle|^2$.
  - In the case of degeneracy (where multiple eigenstates have the same eigenvalue) the probability of measuring said eigenvalue is the sum of the norm-squares over all degenerate eigenstates. So the probability of measuring eigenvalue q is $P_q = \sum_j |c_j|^2 = \sum_j |\langle f_j|\Psi\rangle|^2$ where the summation in j is over all eigenstates that share the eigenvalue q.
- After measurement, the ket state "collapses" to the eigenstate that corresponds to the measured eigenvalue. Meaning, if you measure q_4 as your result, after the measurement the ket state is $|\Psi_{\text{after}}\rangle = |f_4\rangle$, no matter what the ket state was before the measurement.
  - In the case of degeneracy, measuring a given eigenvalue collapses the ket state to a superposition of all of the eigenstates that share the measured eigenvalue with the relative probability amplitude for each eigenstate being proportional to the probability amplitude of that eigenstate in the original ket state. This means that

$$|\Psi_{\text{after}}\rangle = \frac{\sum_j \langle f_j|\Psi_{\text{before}}\rangle |f_j\rangle}{\sqrt{\sum_j |\langle f_j|\Psi_{\text{before}}\rangle|^2}}$$

where the summation in j is over all eigenstates that share the eigenvalue that was observed in the measurement.

- The previous definitions have dealt with hermitian operators with discrete spectra (meaning that the set of eigenstates is countable). All of these rules can be extended to hermitian operators with continuous spectra (such as $\hat{p}$ or $\hat{x}$). For these operators we will say that $\hat{Q}|f_q\rangle = q|f_q\rangle$ where the eigenvalues q are a continuous variable.
  - The eigenstates are no longer actually normalizable, but they will be Dirac orthonormal: $\langle f_p|f_q\rangle = \delta(p-q)$.
  - The eigenstates are still complete. Any ket state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \int c(q)\,|f_q\rangle\,dq \quad \text{where} \quad c(q) = \langle f_q|\Psi\rangle$$

  - $|c(q)|^2$ is now a probability density. The probability of measuring a value between q and $q+dq$ is

$$P = |c(q)|^2\,dq = |\langle f_q|\Psi\rangle|^2\,dq$$

Example problem

Suppose that you have a two-state system (such as a spin-1/2 particle). This system has a Hamiltonian $\hat{H}$ and an observable $\hat{Q}$. In some basis they are given by

$$\hat{H} \to \hbar\omega_0 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \hat{Q} \to q_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

At time $t = 0$ the ket state of this particle in the same basis is given by

$$|\Psi(t=0)\rangle \to \begin{pmatrix} 0.8 \\ 0.6i \end{pmatrix}$$

You make a measurement of the observable Q at time t. What are the possible outcomes and their probabilities? What is the expectation value $\langle Q \rangle$?

Step 1: Determine the eigenvectors and eigenvalues of the matrices.

For the Hamiltonian: $\begin{vmatrix} 0-\lambda & i \\ -i & 0-\lambda \end{vmatrix} = (\lambda^2 - 1) = 0$. So $\lambda = \pm 1$ and the energy eigenvalues are $\pm\hbar\omega_0$.

First eigenvector, for $E_1 = -\hbar\omega_0$: $\hbar\omega_0 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \hbar\omega_0 \begin{pmatrix} i\beta \\ -i\alpha \end{pmatrix} = -\hbar\omega_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Thus, $\beta = i\alpha$. Upon normalization we have $\boxed{|s_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}}$.

Second eigenvector, for $E_2 = +\hbar\omega_0$: $\hbar\omega_0 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \hbar\omega_0 \begin{pmatrix} i\beta \\ -i\alpha \end{pmatrix} =$

$+\hbar\omega_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Thus, $\beta = -i\alpha$. Upon normalization we have $\boxed{|s_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}}$.

For the other observable, $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 0$, So $\lambda = 0$ or 2 and the eigenvalues are 0 and $2q_0$.

First eigenvector, for $q_1 = 0$: $q_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = q_0 \begin{pmatrix} \alpha+\beta \\ \alpha+\beta \end{pmatrix} = 0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Thus $\beta = -\alpha$. Upon normalization we have $\boxed{|v_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$.

Second eigenvector, for $q_2 = 2q_00$: $q_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = q_0 \begin{pmatrix} \alpha+\beta \\ \alpha+\beta \end{pmatrix} = 2q_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Thus $\beta = \alpha$. Upon normalization we have $\boxed{|v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

<u>Step 2</u>: Write the initial ket state in terms of the energy eigenstates, then determine the time-dependent ket state.

$$|\Psi(t=0)\rangle \rightarrow \begin{pmatrix} 0.8 \\ 0.6i \end{pmatrix} = c_-|s_-\rangle + c_+|s_+\rangle.$$

$$c_- = \langle s_-|\Psi(t=0)\rangle = \tfrac{1}{\sqrt{2}}(1 \quad i)^* \begin{pmatrix} 0.8 \\ 0.6i \end{pmatrix} = \tfrac{1}{\sqrt{2}}(1 \quad -i) \begin{pmatrix} 0.8 \\ 0.6i \end{pmatrix} = \frac{1.4}{\sqrt{2}}$$

$$c_+ = \langle s_+|\Psi(t=0)\rangle = \tfrac{1}{\sqrt{2}}(1 \quad -i)^* \begin{pmatrix} 0.8 \\ 0.6i \end{pmatrix} = \tfrac{1}{\sqrt{2}}(1 \quad i) \begin{pmatrix} 0.8 \\ 0.6i \end{pmatrix} = \frac{0.2}{\sqrt{2}}$$

$$|\Psi(t)\rangle = \frac{1.4}{\sqrt{2}}|s_-\rangle e^{-i(-\hbar\omega_0)t/\hbar} + \frac{0.2}{\sqrt{2}}|s_+\rangle e^{-i(\hbar\omega_0)t/\hbar}$$
$$= \frac{1.4}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{+i\omega_0 t} + \frac{0.2}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\omega_0 t}$$

$$\boxed{|\Psi(t)\rangle \to \begin{pmatrix} 0.7e^{+i\omega_0 t} + 0.1e^{-i\omega_0 t} \\ 0.7ie^{+i\omega_0 t} - 0.1ie^{-i\omega_0 t} \end{pmatrix}}$$

Step 3: Write the time-dependent ket state in terms of the observable eigenstates.

$$|\Psi(t)\rangle \to \begin{pmatrix} 0.7e^{+i\omega_0 t} + 0.1e^{-i\omega_0 t} \\ 0.7ie^{+i\omega_0 t} - 0.1ie^{-i\omega_0 t} \end{pmatrix} = c_0(t)|v_0\rangle + c_2(t)|v_2\rangle$$

$$c_0(t) = \langle v_0|\Psi(t)\rangle = \tfrac{1}{\sqrt{2}}(1 \quad -1)^* \begin{pmatrix} 0.7e^{+i\omega_0 t} + 0.1e^{-i\omega_0 t} \\ 0.7ie^{+i\omega_0 t} - 0.1ie^{-i\omega_0 t} \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left[(0.7 - 0.7i)\, e^{+i\omega_0 t} + (0.1 + 0.1i)\, e^{-i\omega_0 t}\right]$$

$$c_2(t) = \langle v_2|\Psi(t)\rangle = \tfrac{1}{\sqrt{2}}(1 \quad 1)^* \begin{pmatrix} 0.7e^{+i\omega_0 t} + 0.1e^{-i\omega_0 t} \\ 0.7ie^{+i\omega_0 t} - 0.1ie^{-i\omega_0 t} \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left[(0.7 + 0.7i)\, e^{+i\omega_0 t} + (0.1 - 0.1i)\, e^{-i\omega_0 t}\right]$$

Step 4: Determine the probabilities and/or expectation values.

If you measure observable Q, the answer must be an eigenvalue of $\hat{Q}$: either 0 or $2q_0$.

The probability of measuring 0 is $P_0(t) = |c_0(t)|^2 = c_0 c_0^*$.

$$\begin{aligned}
P_0(t) =& \frac{1}{\sqrt{2}}\left[(0.7-0.7i)\,e^{+i\omega_0 t} + (0.1+0.1i)\,e^{-i\omega_0 t}\right]\frac{1}{\sqrt{2}}\left[(0.7+0.7i)\,e^{-i\omega_0 t} + (0.1-0.1i)\,e^{+i\omega_0 t}\right] \\
=& \frac{1}{2}\left[|0.7-0.7i|^2 + (0.7-0.7i)(0.1-0.1i)e^{+i2\omega_0 t} + (0.1+0.1i)(0.7+0.7i)e^{-i2\omega_0 t} + |0.1+0.1i|^2\right] \\
=& \frac{1}{2}\left[0.98 - 0.14ie^{+i2\omega_0 t} + 0.14ie^{-i2\omega_0 t} + 0.02\right] = \frac{1}{2} - 0.07i\left(e^{+i2\omega_0 t} - e^{-i2\omega_0 t}\right)
\end{aligned}$$

Remember: for real θ, $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ while $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$.

Thus $\boxed{P_0(t) = 0.5 + 0.14\sin(2\omega_0 t)}$ is the probability of measuring 0.

The probability of measuring $2q_0$ is $P_2(t) = |c_2(t)|^2 = c_2 c_2^*$.

$$\begin{aligned}
P_2(t) =& \frac{1}{\sqrt{2}}\left[(0.7+0.7i)\,e^{+i\omega_0 t} + (0.1-0.1i)\,e^{-i\omega_0 t}\right]\frac{1}{\sqrt{2}}\left[(0.7-0.7i)\,e^{-i\omega_0 t} + (0.1+0.1i)\,e^{+i\omega_0 t}\right] \\
=& \frac{1}{2}\left[|0.7+0.7i|^2 + (0.7+0.7i)(0.1+0.1i)e^{+i2\omega_0 t} + (0.1-0.1i)(0.7-0.7i)e^{-i2\omega_0 t} + |0.1-0.1i|^2\right] \\
=& \frac{1}{2}\left[0.98 + 0.14ie^{+i2\omega_0 t} - 0.14ie^{-i2\omega_0 t} + 0.02\right] = \frac{1}{2} + 0.07i\left(e^{+i2\omega_0 t} - e^{-i2\omega_0 t}\right)
\end{aligned}$$

Thus $\boxed{P_2(t) = 0.5 - 0.14\sin(2\omega_0 t)}$ is the probability of measuring $2q_0$.

$\langle Q\rangle = \langle\Psi(t)|\hat{Q}|\Psi(t)\rangle$.

$$\begin{aligned}
\langle Q\rangle =& \begin{pmatrix} 0.7e^{+i\omega_0 t} + 0.1e^{-i\omega_0 t} & 0.7ie^{+i\omega_0 t} - 0.1ie^{-i\omega_0 t}\end{pmatrix}^* q_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0.7e^{+i\omega_0 t} + 0.1e^{-i\omega_0 t} \\ 0.7ie^{+i\omega_0 t} - 0.1ie^{-i\omega_0 t}\end{pmatrix} \\
=& \begin{pmatrix} 0.7e^{-i\omega_0 t} + 0.1e^{+i\omega_0 t} & -0.7ie^{-i\omega_0 t} + 0.1ie^{+i\omega_0 t}\end{pmatrix} q_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0.7e^{+i\omega_0 t} + 0.1e^{-i\omega_0 t} \\ 0.7ie^{+i\omega_0 t} - 0.1ie^{-i\omega_0 t}\end{pmatrix} \\
=& q_0 \begin{pmatrix} 0.7e^{-i\omega_0 t} + 0.1e^{+i\omega_0 t} & -0.7ie^{-i\omega_0 t} + 0.1ie^{+i\omega_0 t}\end{pmatrix}\begin{pmatrix} 0.7(1+i)e^{+i\omega_0 t} + 0.1(1-i)e^{-i\omega_0 t} \\ 0.7(1+i)e^{+i\omega_0 t} + 0.1(1-i)e^{-i\omega_0 t}\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
\langle Q\rangle = q_0\Big[&(0.7e^{-i\omega_0 t}+0.1e^{+i\omega_0 t})(0.7(1+i)e^{+i\omega_0 t}+0.1(1-i)e^{-i\omega_0 t})\\
&+(-0.7ie^{-i\omega_0 t}+0.1ie^{+i\omega_0 t})(0.7(1+i)e^{+i\omega_0 t}+0.1(1-i)e^{-i\omega_0 t})\Big]\\
&= q_0[0.7(1-i)e^{-i\omega_0 t}+0.1(1+i)e^{+i\omega_0 t}][0.7(1+i)e^{+i\omega_0 t}+0.1(1-i)e^{-i\omega_0 t}]
\end{aligned}
$$

$$
\begin{aligned}
\langle Q\rangle &= q_0\left[0.49|1-i|^2+0.07(1-i)^2e^{-i2\omega_0 t}+0.07(1+i)^2e^{+i2\omega_0 t}+0.01|1+i|^2\right]\\
&= q_0\left[0.98-0.14ie^{-i2\omega_0 t}+0.14ie^{+i2\omega_0 t}+0.02\right]\\
&= q_0\left[1+0.14i\left(e^{+i2\omega_0 t}-e^{-i2\omega_0 t}\right)\right]
\end{aligned}
$$

$$\boxed{\langle Q\rangle = q_0[1-0.28\sin(2\omega_0 t)]}$$

Note: it is of course also true that $\langle Q\rangle = P_0(t)\cdot 0 + P_2(t)\cdot 2q_0$.