Dirac Notation

We start with a review of Dirac notation, a powerful yet succinct way of writing quantum mechanical mathematical functions. We start by introducing a ket, also called a quantum state vector.

> $|\mathcal{S}\rangle$ is a ket state. It is a normalized vector in complex Hilbert space. It represents the state of a quantum system.

So a ket represents the state of our system. Students will often think of this as similar to the wavefunction; while there are similarities the ket state is more abstract. The wave function of a particle is a function of some variable or variables (position typically). This means that the wave function is written in some basis (again, position). The ket is more general and abstract: it represents the full state fo the system without specifying any particular basis. The wavefunction of a particle is therefore the projection of the key state onto the basis of position, while the momentum space wave function is the projection of this same ket state onto the basis of momentum.

> There is a mathematical operation known as the inner product: $\langle\beta|\alpha\rangle$ represents the inner product of ket $|\beta\rangle$ with ket $|\alpha\rangle$.

This can be thought of as a measure of the overlap between the two states: ”how much of” one state is in the other. The inner product of two kets is a number (similar to how the dot product of two vectors is a scalar). The order matters, as flipping the order will complex conjugate the result: $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.

A ket state is normalized when its inner product with itself is one ($|\alpha\rangle$ is normalized when $\langle\alpha|\alpha\rangle = 1$). Two kets are orthogonal when their inner product is zero ($|\alpha\rangle$ and $|\beta\rangle$ are orthogonal when $\langle\alpha|\beta\rangle = 0$).

In quantum mechanics we are free to choose a basis. In a sense a ket state $|\mathcal{S}\rangle$ is a vector that exists in (Hilbert) space. This vector exists, pointing in some direction, independent of our choice of coordinate system. The basis reflects our choice of coordinate system, with the basis vectors being unit vectors ($\hat{i}$, $\hat{j}$, and $\hat{k}$). A set of basis vectors is complete when any ket state can be written as a linear combination of the basis vectors: a set of vectors $|e_i\rangle$ forms a complete basis when any ket state $|\mathcal{S}\rangle$ can be written as $|\mathcal{S}\rangle = \sum_i c_i|e_i\rangle$. The dimension of our system is the number of independent (orthogonal) basis vectors needed to be complete.

$\langle e_i\|e_i\rangle = 1$	normalization
$\langle e_i\|e_j\rangle = 0$ for $i \neq j$	orthogonality
$\|\mathcal{S}\rangle = \sum_i c_i\|e_i\rangle$	completeness

Operators act on ket states and transform them into other ket states. If I write that $\hat{O}|\alpha\rangle = |\beta\rangle$ this means that an operator ($\hat{O}$) acting on some initial ket state ($|\alpha\rangle$) transforms the state into some other state $|\beta\rangle$. In general these two kets ($|\alpha\rangle$ and $|\beta\rangle$) might be very different from one another, but there are special circumstances where they are the same up to a constant.

Suppose that $\hat{O}|\alpha\rangle = \lambda|\alpha\rangle$: when operator $\hat{O}$ acts on state $|\alpha\rangle$ the result is a constant value λ multiplied by the same state. When this is true we say that $|\alpha\rangle$ is an eigenstate of this operator with eigenvalue λ.

Multiple operators can successively act on a state, and the order that they are applied may well matter. Operators start at the right (closest to the ket) and then act in order moving left. Therefore if we write $\hat{A}\hat{B}\hat{C}|\mathcal{S}\rangle$ the operator $\hat{C}$ acts first on $|\mathcal{S}\rangle$. Then $\hat{B}$ acts second (on the result $\hat{C}|\mathcal{S}\rangle$) and then $\hat{A}$ acts last. We could add in parentheses and think of it as $\hat{A}(\hat{B}(\hat{C}|\mathcal{S}\rangle))$.

$[\hat{A},\, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of operators $\hat{A}$ and $\hat{B}$. If $[\hat{A},\, \hat{B}] = 0$ (so that order does not matter, $\hat{A}\hat{B} = \hat{B}\hat{A}$) we say that these two operators commute.

Let us assume that two operators commute: $[\hat{A},\, \hat{B}] = 0$. Let us further assume that $\hat{A}|\alpha\rangle = \lambda|\alpha\rangle$, meaning that $|\alpha\rangle$ is an eigenstate of operator $\hat{A}$ with eigenvalue λ. We start with the observation that, because these two operators commute, $\hat{A}\hat{B} = \hat{B}\hat{A}$.

$$
\begin{aligned}
\hat{A}\hat{B}|\alpha\rangle &= \hat{B}\hat{A}|\alpha\rangle \\
\hat{A}(\hat{B}|\alpha\rangle) &= \hat{B}(\lambda|\alpha\rangle) \\
\hat{A}(\hat{B}|\alpha\rangle) &= \lambda(\hat{B}|\alpha\rangle)
\end{aligned}
$$

What have we proved? This last lines shows that $\hat{B}|\alpha\rangle$ is **also** an eigenstate of $\hat{A}$ with eigenvalue λ. If we assume that there is no degeneracy (the different eigenstates of $\hat{A}$ all have different eigenvalues) then $\hat{B}|\alpha\rangle$ must be proportional to $|\alpha\rangle$: $\hat{B}|\alpha\rangle = \mu|\alpha\rangle$, meaning that $|\alpha\rangle$ is also an eigenstate of $\hat{B}$. This will true for ant eigenstate of $\hat{A}$, meaning that all eigenstates of $\hat{A}$ are also eigenstates of $\hat{B}$ and vice versa.

(When there is degeneracy, such that multiple eigenstates of at least one operator have the same eigenvalues, then the proof is more complicated. But it will always be possible to construct a set of common eigenstates for the two operators.)

Whenever two operators commute they will share a full set of eigenstates.

If the operators represent physical observables then they are compatible observables. There is no uncertainty principle relating them: if $[\hat{A}, \hat{B}] = 0$ then $\sigma_A \sigma_B \geq 0$. If our system is in a common eigenstates of the two operators then we know with certainty what a measurement of either observable would result in.

We had said previously that $\langle\beta|\alpha\rangle$ represents the inner product of ket $|\beta\rangle$ with ket $|\alpha\rangle$. We also of course have worked with kets by themselves: $|\alpha\rangle$, etc. The key of Dirac notation is that we can also treat the left side of the inner product as a mathematical object called a <u>bra</u>.

$\langle\beta|$ is a mathematical object known as a bra.

What is a bra? This is a harder question to answer. If a ket represents a vector in Hilbert space then a bra represents a linear function of vectors. Or, a bra represents a set of instructions of what one should do to any following ket. In position space (where a ket represents a function) a bra is basically the instruction to complex conjugate then multiply by what follows and then integrate:

$$\langle f| \rightarrow \int f^* \left[\cdots\right] dx$$

in which the ellipses are waiting to be filled by the function of the ket which this bra acts on.

When we turn a ket into a bra, any constants become complex conjugates. Therefore, if $|\beta\rangle = a|e_1\rangle + b|e_2\rangle$ then $\langle\beta| = a^*\langle e_1| + b^*\langle e_2|$.

We have previously describe how an operator acts on a ket to create a new ket: $\hat{O}|\alpha\rangle = |\beta\rangle$. An operator can also act "backwards" on a bra to create a new bra: $\langle\alpha|\hat{O} = \langle\gamma|$. But you may notice that I used different variable names in these two lines: β in $\hat{O}|\alpha\rangle = |\beta\rangle$ and γ in $\langle\alpha|\hat{O} = \langle\gamma|$. That is because the ket $|\beta\rangle$ and the bra $\langle\gamma|$ are not necessarily related to each other.

Operator $\hat{O}^\dagger$ is the Hermitian adjoint of the operator $\hat{O}$. This means that if $\hat{O}|\alpha\rangle = |\beta\rangle$ then $\langle\alpha|\hat{O}^\dagger = \langle\beta|$ for any and all $|\alpha\rangle$.

An operator is said to be Hermitian if it is equal to its own Hermitian adjoint. $\hat{Q}$ is Hermitian if $\hat{Q} = \hat{Q}^\dagger$.

Suppose that $\hat{Q}|\alpha\rangle = \lambda|\alpha\rangle$. This means that $\langle\alpha|\hat{Q}^\dagger = \lambda^*\langle\alpha|$. Now we assume that $\hat{Q}$ is Hermitian: $\hat{Q} = \hat{Q}^\dagger$.

$$
\begin{aligned}
\hat{Q} &= \hat{Q}^\dagger \\
\langle\alpha|\hat{Q}|\alpha\rangle &= \langle\alpha|\hat{Q}^\dagger|\alpha\rangle \\
\langle\alpha|(\hat{Q}|\alpha\rangle) &= (\langle\alpha|\hat{Q}^\dagger)|\alpha\rangle \\
\lambda\langle\alpha|\alpha\rangle &= \lambda^*\langle\alpha|\alpha\rangle \\
\lambda &= \lambda^*
\end{aligned}
$$

Which tells us that λ must be purely real (with no imaginary part).

Hermitian operators have purely real eigenvalues.

Physical observables are associated with Hermitian operators: a phyiscal measurement must have a real result, so only real eigenvalues are allowed.