

# Satellite splat: an inelastic collision with a surface-launched projectile

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## Abstract

A projectile is launched vertically from the surface of an isolated planet. It collides with and sticks to a satellite of equal mass that is initially moving in a circular orbit around the planet. Conservation of mechanical energy, angular momentum, and linear momentum are used to determine whether the combined object will subsequently crash into the surface of the planet, remain in orbit, or escape. The material is appropriate for undergraduate students in an introductory mechanics course.

Keywords: orbital motion, inelastic collision, momentum conservation, energy conservation

## 1. Introduction

Introductory physics courses cover momentum conservation for collisions and the principles of orbital motion for satellites, but seldom combine these ideas in applications. In this paper we present results for a perfectly inelastic collision between two objects moving in the gravitational field of a planet, where one object is initially orbiting the planet and the other is launched from the surface. The analysis is accessible to students familiar with conservation of linear momentum for collisions, and conservation of mechanical energy and angular momentum for orbital motion.

Consider the following problem: a projectile of mass  $m$  is launched vertically with speed  $v_{P0}$  from the surface of an isolated, atmosphere-free, nonrotating spherical planet of radius  $R$  and mass  $M$ . It collides with and sticks to a satellite of equal mass that is initially moving in a circular orbit of radius  $r_S$ . For what ranges of values of  $v_{P0}$  and  $r_S$  will the resulting combined object subsequently crash into the surface of the planet, remain in an elliptical orbit, or escape?

## 2. Relating velocity, mechanical energy, and angular momentum to periapsis

Generalizing Kepler's first law and assuming  $m \ll M$ , all orbital paths of the satellite have the planet's center at the geometrical focus closest to *periapsis*—the point in the orbit where the distance between the planet's center and the satellite is a minimum. For the satellite to crash into the planet, a necessary condition is that this periapsis distance  $r_{\text{peri}} \leq R$ .

For a closed, elliptical orbit, celestial mechanics [1] relates the periapsis distance to the semi-major axis  $a$  and eccentricity  $e$  by  $r_{\text{peri}} = a(1 - e)$ . Instead, here we define an orbit in terms of its conserved mechanical energy and angular momentum. Since all objects accelerate at the same rate in a gravitational field, the *specific* values of these quantities (i.e., divided by the mass of the satellite) define the size and shape of an orbit. If the orbiting object has speed  $v$  when it is a distance  $r$  away from the center of the planet, then its specific mechanical energy is

$$\varepsilon = \frac{1}{2}v^2 - \frac{GM}{r}, \quad (1)$$

where  $G$  is the universal gravitational constant. For bound closed orbits,  $\varepsilon < 0$  and the object's path in space is an ellipse, whereas  $\varepsilon = 0$  results in a parabolic path and  $\varepsilon > 0$  describes a hyperbola (both of which are unbound trajectories) [1, 2].

Since the motions considered here are coplanar, we take the specific angular momentum  $h$  (a vector quantity) to be positive for the satellite's initial orbit (and for any other motion prograde with respect to it) and negative for retrograde motion. Its value can be found from the azimuthal component (orthogonal to the radial component) of the orbital velocity as

$$h = rv_{\theta}. \quad (2)$$

At periapsis, the velocity vector is perpendicular to the position vector, i.e.,  $v_{\theta} = v_{\text{peri}}$  at  $r = r_{\text{peri}}$ . We may then write the specific angular momentum as  $h = r_{\text{peri}} v_{\text{peri}}$  and so from equation (1),

$$\varepsilon = \frac{1}{2} \left( \frac{h}{r_{\text{peri}}} \right)^2 - \frac{GM}{r_{\text{peri}}} \Rightarrow \varepsilon r_{\text{peri}}^2 + GM r_{\text{peri}} - \frac{1}{2} h^2 = 0. \quad (3)$$

This quadratic equation can be solved for  $r_{\text{peri}}$  by taking the smaller positive root. If a larger positive root exists, it corresponds to the maximum (apoapsis) distance for an elliptical orbit with  $\varepsilon < 0$ .

The general path of an object moving in an inverse-square-law gravitational field is a conic section [1, 3]

$$r(\theta) = \frac{h^2}{GM(1 + e \cos \theta)}, \quad (4)$$

where  $e$  is the eccentricity of the orbit. Since  $r = r_{\text{peri}}$  when  $\theta = 0$ , it follows that

$$r_{\text{peri}} = \frac{h^2}{GM(1 + e)}. \quad (5)$$

By substituting this result into equation (3) and rearranging, one finds the standard result [2]

$$e = \sqrt{1 + 2\varepsilon h^2 / (GM)^2}. \quad (6)$$

In the sections that follow we shall use equation (3) to either calculate  $r_{\text{peri}}$  given  $\varepsilon$  and  $h$ , or to set constraints on the orbital paths of interest using the inequality  $r_{\text{peri}} \leq R$  for a postcollision planetary impact. Equations (4) and (6) enable us to plot these orbital paths.

### 3. Adhesive collision between a projectile and a satellite of equal mass

Although  $\varepsilon$  and  $h$  are separately conserved for the projectile and satellite while they are moving freely in the planet's gravitational field, the adhesive collision between them requires us to use the principle of conservation of linear momentum to find their common postcollision velocity. Assume that the projectile has been launched such that it collides with and sticks to the satellite when the projectile's distance from the planet's center first equals  $r_S$ . To analyse the resulting two-dimensional collision at this position, decompose the velocity vectors along the outward radial direction ( $\hat{r}$ ) and the prograde azimuthal direction ( $\hat{\theta}$ ) of the satellite's original orbit. The velocity components of the projectile and satellite just prior to the collision are determined by the specific angular momentum and mechanical energy of each object:  $h_P$  and  $\varepsilon_P$  for the launched projectile, and  $h_S$  and  $\varepsilon_S$  for the satellite. In particular, the azimuthal velocity component of the projectile just prior to impact can be found from its specific angular momentum as

$$v_{P\theta} = \frac{h_P}{r_S}. \quad (7)$$

This result is then combined with the projectile's specific mechanical energy to find its radial velocity component,

$$\varepsilon_P = \frac{1}{2}(v_{Pr}^2 + v_{P\theta}^2) - \frac{GM}{r_S} \Rightarrow v_{Pr} = \sqrt{2\varepsilon_P + \frac{2GM}{r_S} - \frac{h_P^2}{r_S^2}}. \quad (8)$$

These same two equations apply to the satellite's azimuthal and radial velocity components, provided one replaces  $\varepsilon_P$  with  $\varepsilon_S$  and  $h_P$  with  $h_S$ . Just after the collision, the two objects move together with a common velocity of magnitude  $v$ . Conservation of linear momentum determines the postcollision instantaneous orbital velocity vector at  $r=r_S$  of this combined mass as

$$2m\vec{v} = m\vec{v}_P + m\vec{v}_S \Rightarrow \vec{v} = 1/2(v_{P\theta} + v_{S\theta})\hat{\theta} + 1/2(v_{Pr} + v_{Sr})\hat{r}. \quad (9)$$

From these new components we can calculate the postcollision values of  $\varepsilon$  and  $h$  for the combined object using equations (1) and (2). Then we can use equation (3) to derive its new periapsis distance  $r_{\text{peri}}$ . A necessary condition for the object to impact the planet is  $r_{\text{peri}} \leq R$ . We must also ensure that  $\varepsilon < 0$  (i.e., the orbit of the combined object is bound) so that it does not escape before reaching periapsis.

### 4. Case study: a vertically launched projectile colliding with a satellite in a circular orbit

Put yourself in the position of a launch crew tasked with bringing down a satellite moving in a circular orbit of radius  $r_S$ . Your first inclination may be to fire a projectile vertically (i.e., radially) from the planet's surface. The projectile's path is thus a degenerate orbit [4] with periapsis distance  $r_{\text{peri}} = 0$  and specific angular momentum  $h_P = 0$ . There are two questions we shall answer for this case:

1. Over what range of satellite orbital radii would a vertical launch be optimal (in terms of minimizing the launch speed) for bringing the satellite down to the planet's surface?
2. Is there a maximum orbital radius beyond which the satellite cannot be brought down with a vertical launch?

From dimensional arguments, the required projectile launch speed must scale with the surface escape speed  $v_{\text{esc}} = \sqrt{2GM/R}$ , whereas the periapsis of the postcollision orbit and the maximum initial radius of a satellite that can be brought down must both scale with the planet's radius  $R$ . We therefore define the ratio of the launch speed to the escape speed as  $f = v_{\text{p0}}/v_{\text{esc}}$  and the ratio of the satellite's orbital radius to the planet's radius to be  $k = r_{\text{S}}/R$ . For a projectile launched vertically at speed  $v_{\text{p0}}$  from the planet's surface at  $r=R$ , its conserved specific angular momentum and mechanical energy are

$$h_{\text{p}} = 0 \text{ and } \varepsilon_{\text{p}} = \frac{1}{2}v_{\text{p0}}^2 - \frac{GM}{R} = \frac{1}{2}v_{\text{esc}}^2(f^2 - 1). \quad (10)$$

Just before colliding with the satellite at  $r_{\text{S}} = kR$ , the projectile's velocity components are given by equations (7) and (8) as

$$v_{\text{p}\theta} = 0 \text{ and } v_{\text{p}r} = \sqrt{2\varepsilon_{\text{p}} + \frac{2GM}{r_{\text{S}}}} = v_{\text{esc}}\sqrt{f^2 - 1 + \frac{1}{k}}. \quad (11)$$

The satellite is initially in a circular orbit of radius  $r_{\text{S}}$ , so that its velocity components are

$$v_{\text{S}\theta} = \sqrt{\frac{GM}{r_{\text{S}}}} = v_{\text{esc}}\sqrt{\frac{1}{2k}} \text{ and } v_{\text{S}r} = 0. \quad (12)$$

According to equation (9), the velocity components for the combined object just after the collision are

$$v_{\theta} = \frac{v_{\text{S}\theta}}{2} = \frac{v_{\text{esc}}}{\sqrt{8k}} \text{ and } v_r = \frac{v_{\text{p}r}}{2} = \frac{1}{2}v_{\text{esc}}\sqrt{f^2 - 1 + \frac{1}{k}} \quad (13)$$

from equations (11) and (12).

We can now determine  $h$  and  $\varepsilon$  for the combined object's trajectory. From equations (2) and (13), the postcollision specific angular momentum is

$$h = r_{\text{S}}v_{\theta} = \frac{1}{2}\sqrt{GM r_{\text{S}}} = \sqrt{\frac{k}{8}}v_{\text{esc}}R \quad (14)$$

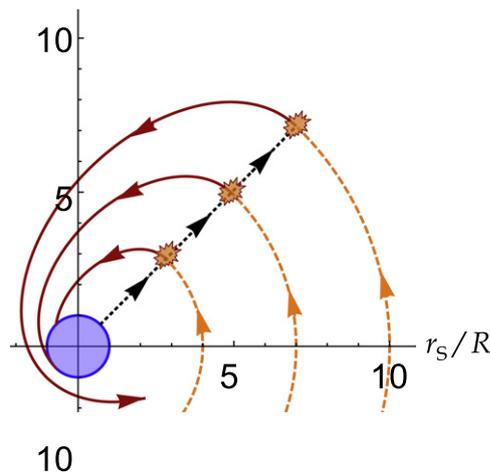
which is half of the satellite's original specific angular momentum, because no net external torque acts on the system of two objects, thereby leaving the total angular momentum  $2mh = mh_{\text{S}}$  unchanged. The new specific mechanical energy is

$$\varepsilon = \frac{1}{2}(v_r^2 + v_{\theta}^2) - \frac{GM}{r_{\text{S}}} = \frac{1}{8}v_{\text{esc}}^2\left(f^2 - 1 - \frac{5}{2k}\right) \quad (15)$$

according to equations (1) and (13). Substituting equations (14) and (15) into (3), the periapsis distance of the combined object is found to be

$$r_{\text{peri}} = \frac{kR}{4 + \sqrt{11 + 2k(f^2 - 1)}} \quad (16)$$

by taking the smaller positive root of the quadratic equation. Another way to get this result is to use equation (5) after substituting equations (14) and (15) into (6) to find  $e$ .



**Figure 1.** Pre- and postcollision paths for three cases of a satellite in a circular orbit (orange dashed lines) that collides and combines with a projectile of equal mass, launched vertically from the surface (black dotted line) at the minimum speed given by equation (17). The satellite orbits shown have  $r_S/R = 4, 7,$  and  $10,$  and the periapsis distances of each postcollision orbit (red solid curves) are given by equation (18).

### 5. Projectile launched vertically at the minimum speed required for a collision

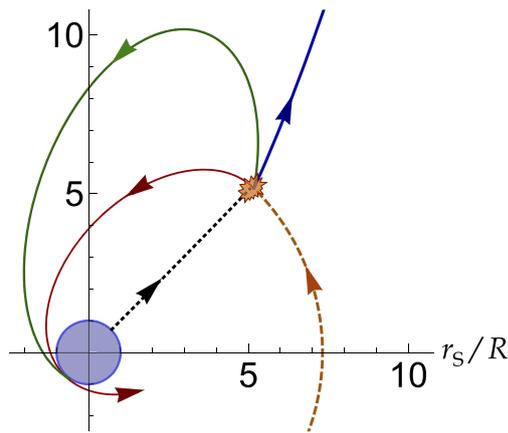
To just reach the satellite, the launch crew can fire the projectile vertically at a minimum speed  $v_{P0\min} = f_{\min} v_{\text{esc}}$  such that the projectile attains apoapsis directly above the launch point at  $r = r_S$ . From equation (11) with  $v_{Pr} = 0$ , the projectile's launch speed must then be

$$v_{P0\min} = \sqrt{2GM \left( \frac{1}{R} - \frac{1}{r_S} \right)} \Rightarrow f_{\min} = \sqrt{1 - \frac{1}{k}} \quad (17)$$

with  $r_S \geq R$  so that  $k \geq 1$ . (The limiting case  $k = 1$  corresponds to a surface-skimming satellite with  $r_S = R$ . In that case  $v_{P0\min} = 0$  because the projectile can be placed on the surface to attach to the satellite as it passes by.) Just before the collision, the projectile's velocity is zero as the satellite runs into it, directly above the launch point. Therefore from conservation of linear momentum, the postcollision velocity of the combined object will be half that of the satellite's original orbital velocity, directed azimuthally, such that the collision point becomes the apoapsis of the combined object's new orbit. The new specific energy can be determined by substituting equation (17) into (15), and  $h$  continues to be given by equation (14). Substituting equation (17) into (16) gives

$$r_{\text{peri}} = \frac{kR}{7} = \frac{r_S}{7}. \quad (18)$$

In order for the combined object to crash into the planet's surface, this result shows that the condition  $r_{\text{peri}} \leq R$  is satisfied if the satellite has an initial orbit of radius  $r_S \leq 7R$ , as illustrated in figure 1. In the limiting case  $r_S = 7R$ , the combined object follows an elliptical orbit that just grazes the planet's surface at periapsis, diametrically opposite the launch point. For that case, the orbital semi-major axis is  $a = \frac{1}{2} \left( r_S + \frac{1}{7} r_S \right)$  and, if desired, Kepler's third law could be used to find the collision-to-impact flight time as half the period [5]. Substituting equations (14) and (18) into (5), we find that the eccentricity of the resulting elliptical path



**Figure 2.** Paths of a projectile launched vertically (dotted black line) which collides and sticks to a satellite in a circular orbit of radius  $r_S = 7.32R$  (dashed orange curve). The solid curves show the postcollision paths of the combined object for three different projectile launch speeds. The red curve is for  $v_{p0} = f_{\min} v_{\text{esc}}$  from equation (17), the green curve is for  $v_{p0} = v_{\text{esc}}$  from equation (19), and the blue curve is for  $v_{p0} = f_{\max} v_{\text{esc}}$  from equation (20).

is  $e = 3/4$ , independent of the satellite's initial orbital radius. Alternatively,  $e$  can be found from  $r_{\text{peri}} = r_S/7 = a(1 - e)$ . Since the launch speed given by equation (17) is the minimum possible to reach the satellite, a vertical launch is optimal for bringing down a satellite in a circular orbit with  $r_S \leq 7R$ .

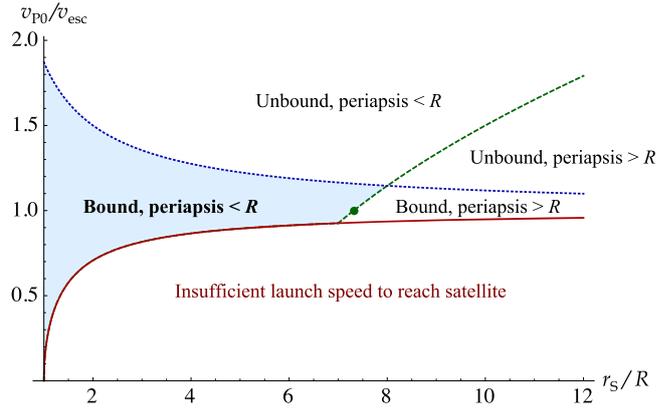
## 6. Effects of increasing the projectile's launch speed

In this section we ask the question: is it possible to bring down a satellite in an orbit larger than  $7R$  with a vertically launched projectile by increasing the launch speed? The projectile hits the satellite from below and pushes it outward. The postcollision specific angular momentum remains given by equation (14). By way of example, suppose the projectile is launched at the escape speed. We can find its periapsis distance from equation (16) with  $f = 1$  as

$$r_{\text{peri}} = \frac{kR}{4 + \sqrt{11}} \approx \frac{kR}{7.32}, \quad (19)$$

and equation (15) confirms that  $\varepsilon < 0$ . Consequently this escape-speed projectile can bring down an equal-mass satellite in a circular orbit with  $r_S \leq 7.32R$ . The extreme case is plotted as the green curve in figure 2.

The red curve shows that if the projectile has zero radial speed when it collides with a satellite at  $r_S = 7.32R$ , it will not quite distort the satellite's orbit sufficiently to bring it all the way down to the planet's surface. If we increase the projectile's launch speed beyond  $v_{\text{esc}}$  it will then have  $\varepsilon_P > 0$ . Even though some mechanical energy is lost in the collision, it may still be possible for the combined object to escape. We must ensure that  $\varepsilon < 0$  after the collision if we desire to bring it down to the planet's surface. From equation (15), this requirement implies that the projectile's launch speed must obey



**Figure 3.** Effect of the projectile's launch speed and the satellite's initial orbital radius on the postcollision path of the combined object. The blue shaded region spans the initial conditions that result in the combined object impacting the planet's surface. The solid red curve plots the minimum launch speed to reach the satellite, from equation (17). The dotted blue line indicates the maximum launch speed for a bound postcollision orbit, according to equation (20). The dashed green line gives the launch speed for which the postcollision periapsis distance is  $R$ , from equation (21). The point marked by a dot at  $r_s/R=7.32$  and  $f=1.0$  corresponds to the launch speed and radius that result in the green postcollision orbit in figure 2.

$$f < f_{\max} = \sqrt{\frac{2k+5}{2k}}. \quad (20)$$

This function is plotted as the dotted blue curve in figure 3, which divides the diagram into regions of bound and unbound motion of the combined object. For launch speeds and satellite radii on this curve, the combined object moves away from the planet on a parabolic escape trajectory. For example, the blue curve in figure 2 shows such a trajectory.

For  $r_s > 7R$ , not all bound postcollision orbits will impact the planet's surface (e.g., the red orbit in figure 2). To determine the minimum launch speed  $v_{P0R} = f_R v_{\text{esc}}$  necessary to cause the combined object to impact the planet's surface, solve equation (16) with  $r_{\text{peri}} \leq R$  to find

$$f \geq f_R = \sqrt{\frac{(k-1)(k-5)}{2k}}. \quad (21)$$

This launch speed is plotted as the green dashed curve in figure 3, but it is not drawn for values of  $r_s/R < 7$  because the projectile would not reach the satellite for those values. That green curve divides the diagram into two regions, with postcollision periapsis distances less than or greater than  $R$ . Figure 3 shows that there is an absolute maximum radius of the satellite's orbit, beyond which it cannot be brought down to the planet's surface for *any* vertical launch speed of an equal-mass projectile. The limit occurs when  $f_R = f_{\max}$ . Equating the right-hand sides of equations (20) and (21) gives the point of intersection of the two curves,

$$f_R = f_{\max} \Rightarrow k = 8 \text{ and } f = \frac{\sqrt{21}}{4} \approx 1.15. \quad (22)$$

That is, a projectile launched vertically from the planet's surface can bring down an equal-mass satellite as long as  $r_s < 8R$ . More generally, the satellite can be brought down to the

planet's surface with an equal-mass vertically launched projectile, provided that its initial orbital radius and the projectile's launch speed lie in the region of figure 3 labelled 'bound, periapsis  $< R$ '.

The existence of such a maximum radius makes sense intuitively if one considers the effect of the collision on the satellite. As  $r_S$  is increased, the satellite's angular momentum  $mh_S$  (a measure of 'motion around' the planet) increases, even as its orbital speed decreases. Although  $h_S$  is always halved by the collision, the increase with  $r_S$  makes it increasingly difficult to reduce the periapsis distance to less than  $R$ . The only way to do so is for the collision to increase the satellite's radial motion, as for the green curve in figure 2, such that it subsequently moves inward rapidly towards small radii. For  $r_S < 8R$ , there is a range of possible projectile speeds that can accomplish that, indicated by the shaded region in figure 3. Eventually, however, the satellite's initial radius and angular momentum become so large that the necessary addition to its radial speed in order to reduce its periapsis causes it to escape.

Imparting a radial impulse to a satellite in a circular orbit has no effect on its orbital angular momentum, but increases its mechanical energy. For more advanced students, this fact can be used to provide an example of an orbiting object's effective potential and the 'centrifugal barrier' [6].

## 7. Concluding remarks

For  $r_S > 7R$ , a vertical launch is no longer optimal in terms of minimizing the launch speed to bring a satellite down to the planet's surface. Instead, the projectile should be launched with some azimuthal velocity retrograde to the satellite's orbit, to create a head-on collision. That will reduce the system's angular momentum, and thereby decrease its periapsis distance according to equation (3). For such nonvertical launches, there exists a maximum radius from which the satellite can be brought down which is larger than the value  $8R$  found here for a vertical launch.

Possible applications of inelastic collisions in space include orbital debris removal [7] and asteroid deflection [8]. Astronomers are also interested in the role of inelastic collisions in the evolution of solar systems [9], planetary rings [10], and protoplanetary discs [11].

Students can explore the effects of varying the mass ratio of the projectile and satellite, and can calculate the launch and impact times. Our hope is that the present paper will inspire further investigations and help introduce students to the fascinating field of *astrodynamics*, defined as 'the application of Newtonian mechanics to manmade objects in space such as rockets and spacecraft.'

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## References

- [1] Bate B, Mueller D and White J 1971 *Fundamentals of Astrodynamics* (New York: Dover)
- [2] Iona M 1963 Satellite orbits *Phys. Teach.* **1** 55–62
- [3] Mungan C E 2005 Another comment on 'eccentricity as a vector' *Eur. J. Phys.* **26** L7–9
- [4] Mungan C E 2009 Radial motion of two mutually attracting particles *Phys. Teach.* **40** 7 502–7
- [5] Noll E D 1996 Kepler's third law for elliptical orbits *Phys. Teach.* **34** 42–3

- [6] Bucher M 1994 Geometric correspondence between effective potential and orbit of Kepler motion *Am. J. Phys.* **62** 373–4
- [7] Bacon J B 2009 Sizing of mother ship and catcher missions for LEO small debris and for GEO large object capture *Int. Conf. for Orbital Debris Removal* <http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/20090041616.pdf>
- [8] Cheng A F 2013 AIDA: test of asteroid deflection by spacecraft impact *44th Lunar and Planetary Science Conf.* [www.lpi.usra.edu/meetings/lpsc2013/pdf/2985.pdf](http://www.lpi.usra.edu/meetings/lpsc2013/pdf/2985.pdf)
- [9] Mohazzabi P and Luecke J A 2003 Asteroid impact and eccentricity of Earth's orbit *Am. J. Phys.* **71** 687–90
- [10] Tiscareno M S *et al* 2013 Observations of ejecta clouds produced by impacts onto Saturn's rings *Science* **340** 460–4
- [11] Rafikov R R 2004 Fast accretion of small planetesimals by protoplanetary cores *Astron. J.* **128** 1348–63