Levitating a strip of paper by blowing over it

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Abstract

It is shown that if you blow vigorously over a curved strip of paper, it levitates into the shape of a catenary. This result quantifies a common classroom demonstration and is a pedagogically useful addition to other studies of catenaries in an intermediate classical mechanics course.

Keywords: Bernoulli’s principle, centripetal acceleration, physics demonstrations, catenary, Coandă effect

A common demonstration consists in blowing hard over the top surface of a curved strip of paper held with an end pressed against one’s chin just below one’s lower lip. The strip is then observed to rise. This effect is often explained in terms of Bernoulli’s principle, which relates an increase in the speed of a fluid at some point along a horizontal streamline to a drop in the driving pressure at the same point \cite{1}. However, that explanation is flawed because a streamline of air above the strip originates in the blower’s lungs and cannot be directly compared to air below the strip at atmospheric pressure \cite{2}. As a convincing counter-example, if the strip is held initially vertical, blowing along either side does not cause the strip to systematically deflect toward the air stream \cite{3}. (Readers are invited to immediately try these two experiments with a 4 cm × 20 cm strip of light-weight paper to see for themselves!) So why then does the curved strip get deflected by blowing above it? The answer is that the flowing air is entrained to follow the downward curve of the strip by the Coandă effect \cite{4–6}. Since the strip pulls the air downward, then the air pulls upward on the curved strip according to Newton’s third law. (A similar explanation holds if you blow under the strip, as the paper pushes the air downward, which you can again demonstrate. However, that effect is much less surprising to students.)
Specifically, suppose a strip of paper is clamped horizontally at one end between the fingers of your hand, exerting a tension $T_0$ on that end in the negative $x$ direction, as sketched in figure 1. You blow along the top surface of the strip, starting from where you are holding it, such that the air moves at approximately the same speed $v$ along its entire length. Consider an arbitrary point along the strip. At this point, the tension $T$ in the paper makes an angle $\alpha$ with the horizontal. Suppose the paper has thickness $t$ and volumetric mass density $\rho$, and the strip has width $w$ and total length $L$. The paper is assumed to be flexible enough that its transverse rigidity can be neglected. (That means the strip will initially have to be draped across your other hand so that it adopts a curved shape.)

The force $dF$ due to the pressure difference normal to an element of the paper of length $ds$ at the point of interest is given by Newton’s second law in terms of the centripetal acceleration [3] as

$$dF = \rho w t u^2 = \frac{Ky''}{(1 + y'^2)^{3/2}} ds,$$

where $r$ is the radius of curvature [7] at the point and primes indicate $x$ derivatives. Here the constant $K$ is equal to $\rho w t v^2 = \lambda v^2$ where $\lambda$ is the linear mass density of the strip of paper.

The horizontal component of this force is

$$dF_x = \frac{Ky''}{(1 + y'^2)^{3/2}} ds \sin \alpha = \frac{K y''}{(1 + y'^2)^{1/2}} dy = \frac{K y''}{(1 + y'^2)^{3/2}} dx.$$

Thus the total horizontal component of the blowing force along the section of paper from where you are holding it (defining the origin) to the point $(x, y)$ is

$$F_x = \int_0^x \frac{K y''}{(1 + y'^2)^{3/2}} dx = K - \frac{K}{\sqrt{1 + y'^2}} \left(\frac{1}{x}ight),$$

so that

$$T_0 = T \cos \alpha + K - \frac{K}{\sqrt{1 + y'^2}}.$$

Likewise the vertical component of the blowing force on the element is

$$dF_y = -\frac{K y''}{(1 + y'^2)^{3/2}} dx.$$
(noting that downwards has been chosen as +y in figure 1) which integrates as

\[ F_y = -\int_0^x \frac{K_y''}{(1 + y'^2)^{3/2}} \, dx = -\frac{K_y'}{\sqrt{1 + y'^2}}. \]  

(6)

Given that the weight of the section of length \( s \) extending from the origin to the point of interest is \( W = \lambda gs \), then

\[ \lambda gs + T \sin \alpha = \frac{K_y'}{\sqrt{1 + y'^2}} = \frac{K \tan \alpha}{\sqrt{1 + y'^2}}. \]  

(7)

Solve the left-hand side of equation (7) for \( \sin \alpha \) and the right-hand side of equation (4) for \( \cos \alpha \), and divide those two results to get

\[ \tan \alpha = \frac{L}{L(k - 1)} \approx \frac{\Lambda}{Lk} s \]  

(8)

using the two dimensionless constants \( k \equiv \lambda u^2/T_0 \) and \( \Lambda \equiv \lambda gL/T_0 \). The approximate equality in the second step of equation (8) assumes that one blows hard enough that \( u \gg \sqrt{T_0/\Lambda} \). Differentiating equation (8) with respect to \( x \) gives

\[ \sec^2 \alpha \frac{d\alpha}{dx} = \frac{\Lambda}{Lk} \sec \alpha. \]  

(9)

Separating and integrating this differential equation results in

\[ \frac{\Lambda}{Lk} x = \int_0^\alpha \sec \alpha \, d\alpha = \ln(\tan \alpha + \sqrt{1 + \tan^2 \alpha}). \]  

(10)

Solve this equation for \( \tan \alpha = \frac{dy}{dx} \), and separate and integrate again to show that the strip of paper adopts the shape of a catenary,

\[ Y = \cosh(X) - 1 \]  

(11)

in terms of the dimensionless coordinates \( X \equiv gx/v^2 \) and \( Y \equiv gy/v^2 \). A heavy cable hanging from two fixed points and the curve of revolution describing a soap film suspended between two coaxial circular rings also have this characteristic shape [8]. For vigorous blowing, this final equation (11) is independent of the end tension \( T_0 \). That independence no longer holds for weak blowing, in which case the shape of the strip depends on the transverse rigidity of the paper.
As a specific example, suppose that $v = 1.00 \text{ m s}^{-1}$, $g = 9.80 \text{ m s}^{-2}$, and $L = 10.2 \text{ cm}$ so that $gL/v^2 = 1$. Substituting equation (8) into (10) and evaluating the result at $s = L$ implies that the value of $X$ runs from 0 to $\ln(\sqrt{2} + 1) \approx 0.881$ along the length of the strip. Equation (11) then implies that $Y$ increases from 0 to $\sqrt{2} - 1 \approx 0.414$. Figure 2 presents a to-scale graph of the resulting shape of the paper.

References