Coefficient of performance of Stirling refrigerators

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Abstract

Stirling coolers transfer heat in or out of the working fluid during all four stages of their operation, and their coefficient of performance depends on whether the non-isothermal heat exchanges are performed reversibly or irreversibly. Both of these possibilities can in principle be arranged. Notably, if the working fluid is an ideal gas, the input of energy in the form of heat during one isochoric step is equal in magnitude to the output during the other isochoric step in the cycle. The theoretical performance of the fridge can then attain the reversible Carnot limit if a regenerator is used, which is a high heat capacity material through which the gas flows. Various Stirling refrigerator configurations are analysed in this article at a level of presentation suitable for an introductory undergraduate thermodynamics course.

Keywords: Stirling cycle, refrigerator, coefficient of performance, quasistatic, reversibility, regenerator

1. Introduction

The Stirling cycle, invented in 1816, uses a fixed number of moles $n$ of a working gas (typically air modelled as a diatomic ideal gas) that repeatedly undergoes two isothermal and two isochoric processes [1]. These processes are here assumed to be quasistatic (so that they can be graphed on a pressure–volume diagram) and any dissipation (due to sliding friction or gas viscosity) is neglected. However, the heat transfers during the two isochoric steps may or may not be thermodynamically reversible, as will be discussed later. The cycle can be run as a valveless engine from the heat input of a flame applied to one end of the gas cylinder, the other end of which is ambiently cooled by fins, and between the two ends of which the air is shuttled back and forth using a displacer [2]. Alternatively, if the flywheel is driven by an electric motor, the cycle can be run as a refrigerator and used to cool samples down starting from room temperature [3]. To efficiently perform the isochoric compression and expansion
of the gas, a regenerator is used which is a heat exchanger through which the hot and cold gas alternately flows from one end of the cylinder to the other. Classroom demonstration units often use metal wool [2, 3] or porous foam [4] as the regenerator, whereas research devices may use an array of closely spaced metal plates [5]. The development of regenerators of simultaneously low fluid flow resistance and high heat capacity [6] is a key to the success of modern Stirling refrigerators [7], used to condense gases and to cool infrared sensors [8].

The goal of this article is to analyse the coefficient of performance (COP) of a Stirling refrigerator with and without a regenerator, both to demonstrate the practical importance of that exchanger and as an example of how the COP of a fridge depends on whether or not the heat transfers to the working fluid are reversible. Further, it has recently been asserted that it is impossible to construct a thermodynamic cooling cycle that lacks adiabatic steps [9], but the present analysis is a counter-example. Introductory physics textbooks typically only analyse refrigerator cycles that include explicit adiabatic stages. Specifically, almost all explore the Carnot refrigerator, and some also discuss Brayton or Otto coolers. In contrast, the Stirling refrigerator is woefully neglected in such books and it is hoped the present paper will help remedy this gap.

In all cases, the Stirling refrigerators treated here have exactly two external temperature reservoirs: a hot one at absolute temperature $T_H$ and a cold bath at $T_C$. Although in principle a sequence of intermediate temperature reservoirs [10] could be used to minimise the irreversibilities during the isochoric heating and cooling steps of the gas in the absence of a regenerator, there is no natural source of such additional reservoirs. (In contrast, the hot and cold reservoirs can be taken to be the ambient surroundings and the cooling compartment of the fridge, respectively.) Creating intermediate-temperature baths by irreversibly mixing together portions of the hot and cold reservoirs has no advantage over directly contacting the gas with the hot and cold baths during the isochoric steps in the manner detailed in section 3 below.

### 2. Reversible regenerative Stirling fridge

The optimal thermodynamic design for a Stirling refrigerator incorporates a regenerator. A schematic of the energy flows in such a fridge is presented in figure 1. Zooming in on the processes undergone by the working gas, figure 2 graphs the details of its pressure $P$ versus volume $V$ during the four stages of the cycle. The energy exhausted as heat from the ideal gas to the hot ambient surroundings during step $ab$ is

$$Q_H = nRT_H \ln \left( \frac{V_a}{V_b} \right),$$

where $R$ is the gas constant. The energy exchanged with the regenerator during steps $bc$ and $da$ is

$$Q_R = \frac{3}{2} nR(T_H - T_C)$$

since the isochoric molar specific heat of a diatomic ideal gas is $c_V = 5R/2$ for temperatures near ambient. Finally the energy withdrawn by heat from the cooling end of the cylinder in step $cd$ is

$$Q_C = nRT_C \ln \left( \frac{V_a}{V_b} \right).$$

In a complete cycle, the input work is $W = Q_H - Q_C$ by energy conservation.

Conceptualise the regenerator as being composed of a set of thermal layers isolated from each other and arranged along the flow axis. (In this respect it is similar to the core of an electromagnetic transformer that is laminated to avoid electrical eddy currents within it. Here
Figure 1. Stirling-cycle cooler having a regenerator.

Figure 2. Quasistatic processes undergone by the working fluid (taken to be a diatomic ideal gas) inside the cylinder of a Stirling refrigerator.
the intent is to avoid thermal currents between the layers of the regenerator.) The layers are arranged with the hottest end initially at temperature $T_h - 2dT$, dropping by $dT$ per layer down to the coldest end at $T_C - dT$. Gas at temperature $T_h$ enters the regenerator (starting at point $b$ in the cycle of figure 2). The gas loses $dQ = 2.5nRdT$ of its internal energy to the first layer (reversibly because the gas and the layer are only infinitesimally different in temperature) such that the gas falls to temperature $T_h - dT$ and the layer rises to temperature $T_h - 2dT$. That occurs if the heat capacity of the layer is equal to that of the gas, $2.5nR$, by appropriate choice of the thickness of the layer. Since the cross-sectional area $A$ of the layer is an independent parameter, this choice of thickness does not restrict the specific heat $c$ of the layer, i.e., $c$ can be made large by choosing the regenerator to have small $A$. A small cross section also makes it easier to establish good thermal contact between the gas and the entire layer. Next, the gas encounters the second layer which is initially at temperature $T_h - 3dT$ and $dQ = 2.5nRdT$ is again transferred so that the gas falls to temperature $T_h - 2dT$ and the second layer rises to $T_h - 2dT$. This process of successive heat transfers continues to the last layer. Gas then exits the regenerator at temperature $T_C$ which matches the final temperature of the last layer of the regenerator.

Assuming every layer can hold its temperature, the subsequent backflow through the regenerator (during process $da$ in figure 2) similarly consists of a sequence of reversible heat transfers. The gas is arranged to be at temperature $T_C - 2dT$ (only infinitesimally lower than $T_C$) when it enters the last layer which is still at $T_C$. Heat will now backtransfer reversibly from the regenerator layer by layer into the gas. Everything done during process $bc$ is undone and one ends up back at the start of the regenerator cycle (except for the gas ending up at $T_h - 2dT$ but that is only negligibly different than the desired $T_h$). Note that the total heat capacity $C$ of the regenerator is $2.5nR$ multiplied by the (large) number of layers, which is thus big in value.

In summary, the temperature of each layer of this ideal regenerator is never more than infinitesimally different than that of the gas flowing through it. That means the heat transfers between the regenerator and the working gas are reversible. Every process in figure 2 can then be made reversible and the COP for this regenerative Stirling fridge\(^1\) can be labelled COP\(_{rev}\). Returning to figure 1, only the heat transfers denoted $Q_H$ and $Q_C$ are with the external temperature reservoirs (connected to the ambient surroundings at temperature $T_h$ and to the cooling load at $T_C$). The regenerator has obviated the need for other reservoirs at intermediate temperatures between $T_h$ and $T_C$ during the isochoric processes [12]. Therefore, calculating the COP as the ratio of the ‘energy transfer wanted’ to the ‘energy transfer paid for’ gives

$$\text{COP}_{\text{rev}} = \frac{Q_C}{W} = \frac{Q_C}{Q_H - Q_C} = \frac{1}{Q_H/Q_C - 1} = \frac{1}{T_h/T_C - 1}$$  (4)

using equations (1) and (3). This result matches the COP of a Carnot refrigerator operating between the same pair of hot and cold temperature reservoirs. That agreement is consistent with the first half of Carnot’s theorem which states that all reversible cycles operating between exactly two external reservoirs of given temperatures have the same efficiency [13].

3. Fully irreversible Stirling refrigerator

Next consider the effect if the regenerator is removed and the Stirling fridge is run without it. The simplest way to accomplish that is to perform each isochoric process irreversibly by

\(^1\) Analysis and measurement of the losses for an actual Stirling device—having a real as opposed to an ideal regenerator—indicate reasonable performance compared to a household Rankine refrigerator [11].
bringing the working gas into contact with a single temperature reservoir. Specifically, during step $bc$ in figure 2, the hot gas is placed into thermal contact with the cold reservoir $T_C$. The change in entropy of the diatomic ideal gas is then

$$\Delta S_{\text{gas}} = \int_{T_{\text{in}}}^{T_{\text{out}}} \frac{n C_V}{T} \, dT = \frac{5}{2} n R \ln \left( \frac{T_C}{T_{\text{in}}} \right)$$

(5)

which is negative, while the gain in entropy of the cold reservoir during step $bc$ is

$$\Delta S_C = \frac{Q_R}{T_C} = \frac{5}{2} n R \left( \frac{T_{\text{in}}}{T_C} - \frac{T_C}{T_{\text{in}}} \right)$$

(6)

using equation (2). It can be shown graphically [14] (and verified numerically by substituting in a value such as $T_{\text{in}} = 1.1 T_C$) that the entropy change of the Universe $\Delta S_{\text{gas}} + \Delta S_C$ is positive for this process, confirming that it is irreversible. Similarly, during step $da$ in figure 2, the cold gas can be placed into thermal contact with the hot reservoir $T_H$. Then equation (5) becomes the positive gain

$$\Delta S_{\text{gas}} = \frac{5}{2} n R \ln \left( \frac{T_H}{T_C} \right),$$

(7)

whereas the hot reservoir has the negative change

$$\Delta S_H = \frac{5}{2} n R \left( \frac{T_C}{T_H} - \frac{T_H}{T_C} \right)$$

(8)

such that the sum $\Delta S_{\text{gas}} + \Delta S_H$ is again positive. Therefore both isochoric processes are irreversible. Nevertheless they can be be made quasistatic by arranging that the heat flow between the gas and the appropriate reservoir is via a sequence of increments of energy $dE$ from one to the other, with time allowed between increments for the gas to reequilibrate. In this manner, the gas is never more than infinitesimally out of thermal equilibrium.

The COP of this irreversible Stirling cycle can be labelled $\text{COP}_{\text{irr}}$. The heat transfers between the gas and cold reservoir are $Q_R$ into the bath during step $bc$ and $Q_C$ out of the reservoir during step $cd$, for a net cooling of the cold reservoir by

$$Q'_C = Q_C - Q_R = n R T_C \ln \left( \frac{V_a}{V_b} \right) - \frac{5}{2} n R \left( T_{\text{in}} - T_C \right)$$

(9)

from equations (2) and (3). Similar reasoning using equations (1) and (2) shows that the net heating of the hot reservoir during steps $da$ and $ab$ is

$$Q'_H = Q_H - Q_R = n R T_H \ln \left( \frac{V_a}{V_b} \right) - \frac{5}{2} n R \left( T_{\text{in}} - T_C \right)$$

(10)

per cycle. To ensure that $Q'_C$ and $Q'_H$ are both positive, a design requirement that

$$0.4 \ln \left( \frac{V_a}{V_b} \right) > \frac{T_{\text{in}}}{T_C} - 1$$

(11)

is imposed, relating the compression ratio to the reservoir temperature ratio [15]. The net work $Q'_H - Q'_C$ remains equal to the same value $W$ as for the regenerative fridge in section 2. Therefore

$$\text{COP}_{\text{irr}} = \frac{Q'_C}{W} = \frac{1}{\frac{T_{\text{in}}}{T_C} - 1} - \frac{2.5}{\ln \left( \frac{V_a}{V_b} \right)}.$$  

(12)

Comparison with equation (4) shows that $\text{COP}_{\text{irr}} < \text{COP}_{\text{rev}}$ because $\ln \left( \frac{V_a}{V_b} \right) > 0$. This result conforms to the second half of Carnot’s theorem stating that any irreversible cycle will have lower efficiency than a reversible cycle if both cycles operate between exactly two external reservoirs of the same temperatures [13]. Just as the COP of a Stirling fridge is reduced if it is operated irreversibly without its regenerator, so too the efficiency of a Stirling heat engine is reduced if it is run irreversibly without a regenerator [16].
Equations (3) and (9) show that the cooling energy per cycle is reduced from \( Q_C \) to \( Q_C' \) by the amount \( Q_R \) that is dumped into the cold reservoir during step \( bc \) in the absence of the regenerator. This cooling reduction can be avoided by running an auxiliary fridge to exhaust \( Q_R \) during step \( bc \) into the hot surroundings where it is of no concern, instead of into the cold sample compartment. A trick is required, however, because \( Q_R \) is not withdrawn from the Stirling cycle at a single gas temperature but rather in the form of increments \( dQ_R \) at temperatures \( T \) that decrease continuously starting with \( T_H \) and ending with \( T_C \). Although infeasible to construct in practice, imagine an infinite sequence of infinitesimal Carnot fridges whose cold fingers are connected to the points running all along path \( bc \) in figure 2. The particular infinitesimal refrigerator connected to the point at which the diatomic gas in the Stirling cycle has temperature \( T \) must withdraw energy \( dQ_R \) at that point, as represented in figure 3. Since the gas volume is constant during stage \( bc \), this infinitesimal heat transfer is given by

\[
dQ_R = \frac{5}{2} n RdT
\]

whose integral from \( T_C \) to \( T_H \) becomes equation (2). The COP of this auxiliary Carnot fridge can be written analogously to equation (4) in two different ways as

\[
\text{COP}_{\text{aux}} = \frac{dQ_R}{dW_{\text{aux}}} = \frac{1}{T_H/T - 1}.
\]
Solving this relation for the infinitesimal work required to run the auxiliary fridge results in
\[ dW_{\text{aux}} = \left( \frac{T_H}{T} - 1 \right) \frac{5}{2} n R dT \] (15)

after substituting equation (13). Finally, instead of an infinite set of such infinitesimal fridges, suppose that a single auxiliary fridge is used whose cold temperature and cooling energy are continuously varied in just this manner. Integrating from \( T_C \) to \( T_H \) then gives the total work required to run this variable auxiliary fridge while cooling the gas inside the Stirling refrigerator along path \( bc \) as
\[ W_{\text{aux}} = 2.5 n R T_C \left( 1 - \frac{T_H}{T_C} + \frac{T_H}{T_C} \ln \frac{T_H}{T_C} \right) \] (16)

The COP of the overall hybrid Stirling–Carnot fridge now becomes
\[ \text{COP}_{\text{hyb}} = \frac{Q_C}{W + W_{\text{aux}}} = \left[ \frac{T_H}{T_C} - 1 + 2.5 \left( 1 - \frac{T_H}{T_C} + \frac{T_H}{T_C} \ln \frac{T_H}{T_C} \right) / \ln \left( \frac{V_a}{V_b} \right) \right]^{-1} \] (17)

This value is intermediate between COP_{\text{irr}} and COP_{\text{rev}}, so that the performance is better than that of the fully irreversible fridge but not as good as that of the fully reversible one. As a numerical example, suppose that \( V_a = 4V_b \) and \( T_H = 1.1T_C \) (corresponding to a 30°C cooling drop near room temperature) which satisfies condition (11). Then equation (4) implies COP_{\text{rev}} = 10, equation (12) gives COP_{\text{irr}} = 8.2, and equation (17) results in COP_{\text{hyb}} = 9.2.

One can further improve the COP by using an auxiliary Carnot engine running between the hot reservoir \( T_H \) and each point at temperature \( T \) along path \( da \) in figure 2 to generate the energy \( dQ_R \) input incrementally to the diatomic gas. This idea is similar to a proposal to efficiently convert heat from a hot gas or wood flame in a house furnace on a winter day to the comparatively low temperature of the room [17]. Such an auxiliary engine generates useful output work which is given by equation (15). This agreement between the work input required to run the auxiliary fridge and the work output from the auxiliary engine is not surprising, because the auxiliary engine becomes precisely the auxiliary fridge running backward when every arrow in figure 3 is reversed in direction. Thus if the Stirling refrigerator is operated in hybrid mode with both a variable auxiliary Carnot fridge along path \( bc \) and a variable auxiliary Carnot engine along path \( da \), then the overall result is a reversible refrigerator with performance COP_{\text{rev}} given by equation (4). For corresponding points at temperature \( T \) along paths \( bc \) and \( da \), the auxiliary fringe and engine have equal and opposite work and heat flows \( dQ_{\text{aux}}, dW_{\text{aux}}, \) and \( dQ_R \) and thus the two auxiliary machines combine to have the same overall effect as the regenerator does in figure 1.

5. Concluding remarks

This paper has explored the performance characteristics of Stirling refrigerators operating between a maximum gas temperature of \( T_H \) and a minimum temperature of \( T_C \). (Similar ideas would apply to Ericsson fridges, in which the isochoric steps are replaced by isobaric stages.) For the fully irreversible case, a Stirling cycle will only function as a fridge (or heat pump) if the natural logarithm of the volume compression ratio \( r \equiv V_a/V_b \) exceeds \( (\gamma - 1)^{-1} (T_H/T_C - 1) \) in value where \( \gamma \) is the adiabatic exponent of the working ideal gas. (Specifically \( \gamma = 5/3 \) for a monatomic gas and \( \gamma = 7/5 \) for a diatomic gas.) In contrast, if \( \ln r < (\gamma - 1)^{-1} (1 - T_C/T_H) \) then the device functions not as a fridge but as the 'cold pump' [9] sketched in figure 4(a) of [15], exhausting heat into the cold reservoir rather than
extracting heat from it. Such a device could find application as a defroster (such as of frozen meat) because it would have a COP $Q_C/(Q_C - Q_H)$ that is higher than that of an electric heater (whose COP = 1). For values of $r$ in the intermediate range $T_C < (\gamma - 1)^{-1}(T_H - T_C) / \ln r < T_H$ a counter-clockwise Stirling cycle gives rise to the inefficient heat pump sketched in figure 4(b) of [15], dumping heat into both the hot and cold reservoirs. That unfortunate situation could occur for a residential heat pump if the outside unit continued to run when the outdoors temperature was colder than the coldest temperature of the working fluid so that heat runs in the undesired direction. (Avoiding such a scenario is the key reason that home heat pumps come with emergency backup electric heaters used on sufficiently cold days.)

In analysing a Stirling refrigerator, it can be helpful to distinguish which parts of the machinery are taken to comprise the ‘system’ and which remaining objects make up the ‘surroundings’. Specifically, for the case of an ideal-gas Stirling fridge having a perfect regenerator, if the system is defined as the gas + regenerator then the two energy transfers labelled $Q_R$ in figure 1 are internal to the system. The cycle then effectively comprises two isothermal steps (ab and cd) and two adiabatic steps (bc and ad). From this perspective, it is not surprising that one obtains the Carnot COP in equation (4). On the other hand if the system is taken to be the gas alone, then at no stage in its operation is the system thermally isolated from the surroundings. The regenerator can then be treated as if it were a single reservoir of periodically varying temperature consistent with the hybrid Stirling–Carnot scheme proposed in section 4. Regardless of what is included in the system, however, the COP must always be given by equation (4) for a reversible Stirling refrigerator. In particular, it does not become equal to $(Q_C + Q_R)/W$ when the system is defined to be the working fluid alone, even though the heat inputs to the gas over the course of a cycle are then $Q_C$ in step cd and $Q_R$ in step da. Only $Q_C$ represents the ‘energy transfer wanted’ and so it alone should be included in the numerator. If the COP depended on how the system boundaries were chosen, then the COP would cease to represent an objective figure of merit for comparing one fridge to another!

Correspondence with Bob Dickerson, Harvey Leff, Dan Schroeder, and John Mottmann has improved the clarity of this article.

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