Orbits of two electrons released from rest in a uniform transverse magnetic field

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Abstract

Two identical charged particles released from rest repel each other radially. A uniform perpendicular magnetic field will then cause their trajectories to curve into a flower petal pattern. The orbit of each particle is approximately circular with a long period for a strong magnetic field, whereas it becomes a figure-eight for a weak magnetic field with each lobe completed in a cyclotron period. For example, such radially bound motions arise for two-dimensional electron gases. The level of treatment is appropriate for an undergraduate calculus-based electromagnetism course.

Keywords: motion of charged particles, crossed electric and magnetic fields, bound orbits, two-dimensional electron gas

(Some figures may appear in colour only in the online journal)

1. Introduction

If two identical charged particles initially separated by a distance $2r_0$ are released from rest then they will mutual repel each other, accelerate, and escape to infinity along diagonally opposite radial paths, with the origin midway between their initial positions. If a uniform external magnetic field of magnitude $B$ is turned on prior to their release in a direction perpendicular to their initial axis of separation, the magnetic force acting on each of them will deflect their trajectories. How large does $B$ have to be to keep the two particles from escaping to infinity? As shown in this article, no matter how weak $B$ is (but provided it is nonzero), the two particles will reach some maximum radial distance $r_{\text{max}}$ from the origin and then symmetrically return toward it, with the two particles instantaneously coming back to rest when they are again separated by $2r_0$. The motion will then repeat, with the two particles continuously orbiting around the origin in the same clockwise or counter-clockwise sense (depending on the sign of their charge for a given direction of the magnetic field) so that they
trace out a ‘flower petal’ pattern. The orbit of each particle will either close back on itself after some integral number of revolutions about the origin, or it will densely cover all points in the plane of motion between \( r_0 \) and \( r_{\text{max}} \) over time without ever retracing itself. Such patterns are an interesting addition to the standard textbook cases of parabolic motion of a charge in a uniform electric field \([1]\), helical motion of a charge in a uniform magnetic field \([1]\), and cycloidal motion of a charge in crossed uniform electric and magnetic fields \([2, 3]\). Numerical exploration of the equations presented in this paper could make for an interesting student project.

A practical realisation of such motion occurs for two-dimensional electron gases in quantum wells, thin semiconductor devices, graphene sheets, or interfaces between certain materials. Application at low temperatures of a strong magnetic field perpendicular to the plane of motion leads to the quantum Hall effect \([4, 5]\). It may not even be necessary to apply an external magnetic field, because the nuclei in certain thin films produce strong fields of their own \([6]\).

Here it is assumed that the classical (i.e. nonquantum and nonrelativistic) laws of motion apply \([7]\). Furthermore, the direct magnetic interaction between the two particles is neglected. (Each moving charge constitutes a current which, according to the Biot–Savart law, creates a magnetic field internal to the system of two particles that adds to the externally applied magnetic field. However, this internal magnetic field falls off rapidly with the inverse-squared distance of separation between the particles. Furthermore, when the particles are close to each other, they move along largely radial paths directly away from or toward each other and thus do not exert a magnetic force on each other.) The resulting motion in this case has been analysed previously using Lagrangian methods with the results expressed in terms of elliptic integrals \([8]\). The purpose of the present article is to use more introductory methods, with a special emphasis on proving that the motion remains bound no matter how small \( B \) is in value.

Intuitively, the reason that the particles cannot escape each other even in the limit of small (but constant) \( B \) is that at large distances their electric interaction gets weak while their speeds grow large\(^1\). Thus eventually the magnetic force becomes sufficiently dominant to turn the particles back toward the origin.

2. Derivation of the equations of motion

By symmetry, the trajectories of the two particles sketched in figure 1 are such that they always remain radially opposite the origin from each other. Thus it is sufficient to compute the path \((r, \phi)\) of only the right-hand particle. Define the \(x\)-axis to point along the initial separation direction of the particles and the \(z\)-axis to be parallel to the external magnetic field. For specificity, assume the two particles are electrons of charge \(-e\) and mass \(m\).

The time derivatives of the plane polar unit vectors are

\[
\frac{d\hat{r}}{dt} = \dot{\phi} \hat{\phi} \quad \text{and} \quad \frac{d\hat{\phi}}{dr} = -\dot{\phi} \hat{r} \quad (1)
\]

using overdots to indicate total time derivatives of scalars. The position of the electron is

\[
\vec{r} = r \hat{r} \quad (2)
\]

\(^1\) As the radial separation between the particles increases, the electric repulsive force always does positive work. Meanwhile the magnetic force does zero work. Thus the kinetic energy of each particle continuously increases as it moves from \( r_0 \) to \( r_{\text{max}} \).
and taking one and then two time derivatives of it respectively gives the velocity and acceleration of the particle as

$$\ddot{v} = \dot{r} \hat{r} + r \ddot{\phi} \hat{\phi}$$  \hspace{1cm} (3)

and

$$\ddot{a} = (\dot{r} - \ddot{\phi}^2 r) \hat{r} + (r \dddot{\phi} + 2 \ddot{\phi} \dot{\phi}) \hat{\phi}$$  \hspace{1cm} (4)

using equation (1). The two forces on the electron are the internal electric repulsion

$$\vec{F}_E = \frac{ke^2}{(2r)^2} \hat{r},$$  \hspace{1cm} (5)

where \( k \) is the Coulomb constant, and the external magnetic force

$$\vec{F}_B = -e\vec{v} \times \vec{B},$$  \hspace{1cm} (6)

where the magnetic field is \( \vec{B} = B \hat{k} \) with constant magnitude and direction. Newton’s second law becomes

$$\vec{F}_E + \vec{F}_B = m \ddot{a}.$$  \hspace{1cm} (7)

Substitute equation (3) into (6) and then that into the left-hand side of (7) together with (5); also substitute equation (4) into the right-hand side of (7). The radial component of the result is

$$\ddot{r} - \dddot{\phi}^2 r = \frac{ke^2}{4mr^2} - \frac{eB \ddot{\phi} r}{m}$$  \hspace{1cm} (8)

and the azimuthal component is

$$r \dddot{\phi} + 2 \ddot{\phi} \dot{r} = \frac{eB \dot{r}}{m}.$$  \hspace{1cm} (9)

These are two coupled differential equations in the two unknowns \( r(t) \) and \( \phi(t) \) that give the orbit as a function of time \( t \). Together with the initial conditions

$$\phi(0) = 0, \ r(0) = r_0, \ \dot{\phi}(0) = 0, \ \text{and} \ \dot{r}(0) = 0,$$  \hspace{1cm} (10)

these coupled equations can be used to calculate the motion of the system.
3. Solution of the equations of motion

The two equations (8) and (9) are second order. Their first integrals determine the angular speed \( \omega \equiv \dot{\phi} \) and radial speed \( V \equiv \dot{r} \). These two speeds can be analytically computed as a function of the radial position \( r \) (rather than of the time \( t \)). That transformation of the independent variable is accomplished using the chain rule. First rewrite the angular acceleration as

\[
\ddot{\phi} = \frac{d}{dr} \left( \frac{d\phi}{dr} \right) = \frac{V}{dr} \frac{dV}{dr}
\]

so that equation (9) becomes

\[
r \frac{d\omega}{dr} + 2\omega = \frac{eB}{m}.
\]

Scaling the radial coordinate by its initial value as \( s \equiv r/r_0 \), equation (12) leads to

\[
\int_0^s \frac{d\omega}{\omega - eB/2m} = -2 \int_1^s \frac{dr}{s}
\]

whose solution is

\[
\omega = \left( \frac{\pi}{T} \right) \left( 1 - s^{-2} \right)
\]

in terms of the cyclotron period\(^2\) \( T = 2\pi m/eB \). As expected, \( \omega \) is positive (corresponding to counter-clockwise revolutions of the electrons about the origin) for all \( s > 1 \Rightarrow r > r_0 \).

Likewise

\[
\ddot{r} = \frac{dr}{dt} \frac{d^2r}{dr^2} = V \frac{dV}{dr}
\]

so that equation (8) can be rewritten as

\[
V \frac{dV}{dr} - \omega^2 r = \frac{ke^2}{4m} s^{-2} - \frac{eB}{m} \omega r.
\]

Scaling the radial speed by the initial position as \( u \equiv V/r_0 = \dot{s} \), equation (16) becomes

\[
u u \frac{du}{ds} - \omega^2 s = \frac{ke^2}{4m r_0^2} s^{-2} - \frac{eB}{m} \omega s
\]

whose variables can be separated using equation (14) and integrated to obtain

\[
u^2 = \left( \frac{\pi}{T} \right)^2 [c(1 - s^{-1}) - (s - s^{-1})^2]
\]

in terms of the dimensionless electric-to-magnetic coupling constant

\[
c = \frac{2mk}{B^2 r_0^2}
\]

The particles attain their maximum radial distance \( r_{\text{max}} \) from the origin when \( u = 0 \). Let the corresponding value of \( s \) at that point be \( s_{\text{max}} \). According to equation (18),

\[
n s_{\text{max}} = (s_{\text{max}} + 1)^2 (s_{\text{max}} - 1).
\]

This equation is to be solved for \( s_{\text{max}} \) subject to the constraints that \( c > 0 \) and \( s_{\text{max}} > 1 \). The right-hand side of equation (20) equals zero when \( s_{\text{max}} = 1 \) and it thereafter monotonically increases with increasing values of \( s_{\text{max}} \). On the other hand, the left-hand side of equation (20)
equals \( c \) when \( s_{\text{max}} = 1 \) and it also monotonically increases with increasing values of \( s_{\text{max}} \), but not as rapidly as the right-hand side (which grows like \( s_{\text{max}}^3 \) in the limit of large \( s_{\text{max}} \)). Thus, as schematically indicated in figure 2, there is one unique positive solution of equation (20) for any given value of \( c \). An analytical formula for this root of the cubic polynomial is

\[
s_{\text{max}} = \frac{2\sqrt{4 + 3c}}{3} \cos \left[ \frac{1}{3} \cos^{-1} \left( \frac{16 - 9c}{2(4 + 3c)^{3/2}} \right) \right] - \frac{1}{3}.
\]  

(21)

Figure 2 shows the expected limiting values \( r_{\text{max}} \to r_0 \) as \( c \to 0 \Rightarrow B \to \infty \) and \( r_{\text{max}} \to \infty \) as \( c \to \infty \Rightarrow B \to 0 \). If \( c \gg 1 \), which occurs for sufficiently small \( B \), it is evident from this figure that \( s_{\text{max}} \gg 1 \). In that limit, equation (20) becomes

\[
\omega s_{\text{max}} \approx s_{\text{max}}^3 \Rightarrow s_{\text{max}} \approx \sqrt[3]{\omega}.
\]

(22)

As \( s \) increases from 1 to \( s_{\text{max}} \), \( u \) is given by the positive square root of equation (18). The ratio of \( \omega \) to \( u \) is equal to

\[
\frac{\omega}{u} = \frac{d\phi}{dt} = \frac{d\phi}{ds}.
\]

(23)

Substituting equations (14) and (18) into this relation and integrating with respect to \( s \) gives the trajectory of the electron as

\[
\phi(s) = \int_1^s \frac{(1 - z^{-2})dz}{\sqrt{c(1 - z^{-1}) - (z - z^{-1})^2}}.
\]

(24)

where the integration variable \( s \) has been rewritten as \( z \) to avoid confusion with the upper limit \( s \). Setting this upper limit to \( s = s_{\text{max}} \) from equation (21) gives the angle \( \phi_{\text{max}} \) that divides the first lobe of the particle’s orbit into two equal halves. Equation (24) only gives the angular coordinate of the trajectory for the first half of the first lobe. That pattern symmetrically repeats itself around the origin for all subsequent half lobes of the orbit. For example, the angular coordinate for the second half of the first lobe is equal to \( 2\phi_{\text{max}} - \phi \) where \( \phi \) is given by equation (24), that for the first half of the second lobe is equal to \( 2\phi_{\text{max}} + \phi \), that for the second half of the second lobe is equal to \( 3\phi_{\text{max}} - \phi \), and so on.

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3 This solution is root \( s_1 \) at [https://en.wikibooks.org/wiki/Trigonometry/The_solution_of_cubic_equations](https://en.wikibooks.org/wiki/Trigonometry/The_solution_of_cubic_equations) where \( A = 1, B = -1 - c, \) and \( C = -1 \).
Similarly \( u = ds/dt \) can be integrated to find the time required for the electron to reach a particular point in the first half of the first lobe of its orbit starting from its release point,

\[
t(s) = \frac{T}{\pi} \int_1^s \frac{dz}{\sqrt{c(1 - z^{-1}) - (z - z^{-1})^2}}.
\] (25)

4. Graphs of the resulting trajectories

Equation (24) can be numerically integrated for any value of \( c \) of interest using a tool such as WolframAlpha\(^4\) to find the angular coordinate \( \phi \) corresponding to a scaled radial coordinate \( s \) between 1 and \( s_{\text{max}} \) in value, where \( s_{\text{max}} \) is given by equation (21). The rectangular coordinates of the electron are then given by

\[
x/r_0 = s \cos \phi \quad \text{and} \quad y/r_0 = s \sin \phi.
\] (26)

In this way, the orbits for three different values of \( c \) have been graphed in figure 3 over enough lobes that the right-hand electron makes approximately one revolution around the origin. The trajectory of the left-hand electron in figure 1 is not shown in figure 3, but it makes exactly the same pattern as the right-hand electron except that it starts at the point \((-1, 0)\) rather than \((1, 0)\). For the smallest value of \( c \) (corresponding to the largest value of \( B \)) as shown in figure 3(a), the orbit has many lobes and \( r \) always remains near \( r_0 \) in value, consistent with the discussion immediately following equation (21).

On the other hand, for the largest value of \( c \) in figure 3, the orbit approximates a pair of circles distributed along the \( y \)-axis. This pattern is explained as follows. According to equation (22), \( s_{\text{max}} \approx \sqrt{c} \) which equals 31.6 for \( c = 1000 \), consistent with the \( y \)-intercepts of the graph. Since \( s \gg 1 \) for most of the trajectory in figure 3(c), the three reciprocal powers of \( z \) in equation (24) can be dropped to get

\[
\phi \approx \int_1^{s_{\text{max}}} \frac{dz}{\sqrt{c - z^2}} \approx \sin^{-1} \frac{s}{\sqrt{c}}.
\] (27)

This formula describes a circle of diameter \( \sqrt{c} \) centred at \((0, \frac{1}{2} \sqrt{c})\) as proven in figure 4. In particular, substituting \( s = s_{\text{max}} \approx \sqrt{c} \) into equation (27) leads to \( \phi_{\text{max}} \approx \pi/2 \) so that the axis of symmetry in figure 3(c) is along the \( y \) direction. Likewise equation (25) shows that the time required for the electron to reach this point of maximum distance from the origin is \( t_{\text{max}} \approx T/2 \) so that the electron executes one circular lobe per cyclotron period, as expected because the two electrons are so far apart from each other for most of their trajectory that they behave like non-interacting charges circling the external magnetic field [9].

5. Conclusions

The trends in figure 3 make it clear that if the external magnetic field is strong (such that \( c \ll 1 \)) then each revolution of an electron about the origin will have a trajectory consisting of a large number of shallow lobes, thus approximating a circle of scaled radius \( s = 1 \). According to equation (14), that implies the angular speed will be small compared to the

\(^4\) For example, type ‘Integrate[(1 - 1/z^2)/Sqrt[10^7(1 - 1/z) - (z - 1/z)^2], \{z, 1, 2.5\}]’ into https://wolframalpha.com/ to find the value of \( \phi \) (in rad) when \( c = 10 \) and \( s = 2.5 \). That corresponds to the point indicated by the green dot in figure 3(b).
Figure 3. Orbits of the right-hand electron in figure 1 for three different values of $c$. The rectangular coordinates $x$ and $y$ are scaled to the initial radius $r_0$, so that the electron starts at the point $(1, 0)$ indicated by the black dot in each panel and revolves counterclockwise about the origin.
cyclotron frequency because the electron will keep starting and stopping as it executes each lobe, so that it only makes slow progress around the origin. On the other hand, when the applied magnetic field is weak (such that $c \gg 1$), then the trajectory of each electron will consist in a figure-eight pattern of two circles distributed vertically about the origin, where each lobe is completed in a cyclotron period and the scaled radius is approximately $\sqrt{e}$.

A weak magnetic field therefore suffices to confine the electrons to within a distance of $r_{\text{max}} \approx (2 k m/r_0)^{1/2} B^{-1}$ from the origin. Since the ratio of the internal electric field between the electrons to the external magnetic field acting on either electron always gets small for large enough distances of separation between the particles, their motion remains bound even if their speeds become relativistic [10]. This conclusion also holds true if one considers the interaction of a large number of electrons (rather than just a pair of them) in a magnetic field, a phenomenon known as ‘dynamic binding’ of a two-dimensional electron gas [11]. The cyclotron period becomes $\gamma T$ where $\gamma$ is the relativistic Lorentz factor [12].

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**References**


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Figure 4. Graph of a circle of diameter $\sqrt{e}$ centred at $(0, \frac{1}{2}\sqrt{e})$. An arbitrary point on the circle is located at scaled polar coordinates $(s, \phi)$. From the geometry of this graph, it follows that $s/2 = \frac{1}{2} \sqrt{e} \cos(\pi/2 - \phi) \Rightarrow s = \sqrt{e} \sin \phi$. 

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