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Calculating the thrust on continuously and discretely emitting rockets

Carl E Mungan*

Volgenau Department of Physics, US Naval Academy, Annapolis, MD 21402, United States of America
* Author to whom any correspondence should be addressed.

E-mail: mungan@usna.edu

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Abstract

Newton's second law written as the rate of change of linear momentum of a system can be re-expressed as a differential equation of motion for a rocket that continuously ejects mass. The result includes a mathematical expression for the thrust force in terms of the rate of mass loss and the relative exhaust velocity. If the rocket is instead propelled by the discrete emission of finite chunks of mass, an analogous equation for the thrust impulse can be deduced, formulated either in terms of the initial or final velocity of the rocket. Some common issues concern how to define thrust and how to write Newton's second law in terms of that thrust and the net external force or impulse acting on the rocket (such as due to gravity and drag). Additionally, a distinction must be made between the center-of-mass velocity and the average velocity for an object of variable mass such as a rocket. The discussion is at a level appropriate for an introductory calculus-based course in physics.

1. Introduction

Rockets can be defined as objects that move in some direction as a result of ejecting mass (either continuously in the form of a fluid or granular material, or discretely such as propelling a boat by throwing bowling balls off the stern) in the opposite direction [1]. Traditionally the motion of a rocket is translational, but it could alternatively be rotational, as in the case of a spinning water sprinkler [2]. Thus, rockets fall into the general category of variable-mass problems [3]. Provided the ejected material has a different velocity than that of the rocket, a term arises in the momentum analysis which is usually called the thrust [4–7] although some care should be taken regarding the nomenclature to avoid creating misconceptions [8]. If an object gains rather than loses mass (but again assuming the accreted material has a nonzero velocity relative to the object), one could call it an inverse rocket. A submarine that fills its ballast tanks with water as it dives [9] is an example of such a system.

This article explores from first principles some key issues associated with the mathematical derivation and calculation of thrust from Newton's laws of motion. As an example of a frequently confusing question, is it always the case that the net external force is equal to $\mathrm{d}\vec{p}/\mathrm{d}t$ for a rocket of momentum $\vec{p}=m\vec{v}$ if both its mass m and velocity $\vec{v}$ are varying with time t? As another example, if a rocket is gaining or losing mass, can the thrust on it nevertheless be zero? After discussing such questions, some interesting applications are suggested below where knowing their answers is helpful.

2. Continuous mass ejection

Consider an object of mass m and velocity $\vec{v}$ (measured in some inertial reference frame S). If m is constant, then Newton's second law is

$$\vec{F}_{\text{ext}}=m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{1}$$

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where $\vec{F}_{\text{ext}}$ is the sum of the external forces on the object. But suppose m is instead variable, as a result of the ejection of mass at velocity $\vec{u}$ as again measured in frame S. Define the 'rocket' to be the main body of interest (because it is carrying astronauts or some other desired payload) of time-varying mass $m(t)$ and velocity $\vec{v}(t)$ such that it has linear momentum $\vec{p}(t) = m\vec{v}$. Define the 'ejecta' to be the mass that has been emitted between some ignition time (at which the clock was set to 0) and the current time t in the form of a continuous stream of mass. The rocket is changing mass at a rate of $\mathrm{d}m/\mathrm{d}t < 0$ and so an infinitesimal ejected element must have mass $-\mathrm{d}m$ in order to be positive[1]. Finally, define the 'system' to be the aggregate of the rocket and ejecta, such that it has constant total mass (equal to the mass m_0 of the rocket at ignition). Let $\vec{F}_{\text{ext}}(t)$ be the net force on the rocket at time t due to agents (such as gravity or drag) *external to the system*; it excludes any interaction force between the rocket and ejecta.

Given these preliminaries, instructors are invited to discuss with their students which of the following statements they think is correct and why? The three options are purposefully chosen to be mutually exclusive but comprehensive, so that exactly one of them is right.

(1) $\vec{F}_{\text{ext}}$ is never equal to $\mathrm{d}\vec{p}/\mathrm{d}t$ while the mass m is changing.

(2) $\vec{F}_{\text{ext}}$ is always equal to $\mathrm{d}\vec{p}/\mathrm{d}t$ according to the generalized form of Newton's second law, which enables one to derive an equation of motion for the rocket.

(3) In some cases $\vec{F}_{\text{ext}}$ is equal to $\mathrm{d}\vec{p}/\mathrm{d}t$, but in other cases they are not equal. Uncertainty about when they are equal is a common source of confusion in analyzing variable-mass problems.

Choice (2) is a common misconception. Some people know that that choice is not correct, and so they swing to the opposite extreme and choose option (1). Actually, the correct answer is (3). To verify that, the starting point (as proven in appendix A) is

$$\vec{F}_{\text{ext}} + \vec{F}_{\text{thrust}} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{2}$$

where the thrust force on the rocket (or simply the 'thrust') is defined as

$$\vec{F}_{\text{thrust}} = \vec{v}_{\text{rel}}\frac{\mathrm{d}m}{\mathrm{d}t} \tag{3}$$

and the velocity of the ejecta relative to the rocket (or simply the 'exhaust velocity') is

$$\vec{v}_{\text{rel}} = \vec{u} - \vec{v} \tag{4}$$

which is often taken to be constant, but need not be so [10]. Standard textbooks [11–14] use these definitions of thrust and exhaust velocity. No mechanism need be specified for the interaction force between the rocket and ejecta; a nonzero thrust can be inferred whenever mass is ejected (so that $\mathrm{d}m/\mathrm{d}t \neq 0$) from an object at a velocity different from that of the object (so that $\vec{u} \neq \vec{v}$). On the flip side, the latter requirement can help prevent misidentification of a thrust when there is actually none. For example, if a loaded railroad car is rolling along a horizontal track and a trapdoor in its floor is opened to release cargo which falls vertically down onto the track, then there is no thrust on the car [15, 16]. Ignoring the external forces of rolling friction and air drag for simplicity, the released cargo (while it is in the air) has the same horizontal velocity $\vec{u}$ that it had just before release when it was traveling with the velocity $\vec{v}$ of the car. As far as the horizontal motion of the system of car plus cargo is concerned, nothing changes just after the trapdoor is opened: both continue to move at velocity $\vec{v}$ along the track. To summarize, equation (1) holds not just if m is constant, but also if mass is released from (or added onto) an object at the same velocity as the object so that $\vec{u} = \vec{v}$.

Returning to the question and three possible responses above, if and only if $\vec{u} = 0$ is it the case that $\vec{F}_{\text{ext}} = \mathrm{d}\vec{p}/\mathrm{d}t$. Therefore choice (3) is the correct option. When $\vec{u} = 0$, equations (2)–(4) can be combined to get

$$\vec{F}_{\text{ext}} - \vec{v}\frac{\mathrm{d}m}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \quad \Rightarrow \quad \vec{F}_{\text{ext}} = \vec{v}\frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}. \tag{5}$$

An example of a rocket problem [17] in which $\vec{u} = 0$ is unrolling a rug along the floor of a house (relative to which the rug does not slip). The inertial frame S is here chosen to be the floor, and the ejected

[1] The rate of mass change $\mathrm{d}m/\mathrm{d}t$ is usually taken to be constant, but it could alternatively be some known function of time [43].

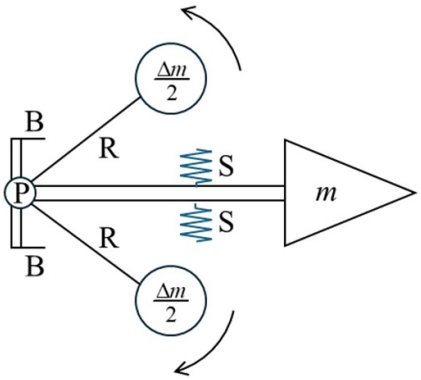


Figure 1. A spring-loaded rocket which ejects total mass Δm leftward to propel a payload of mass m rightward. The system starts out initially at rest with the balls compressed against the springs. Mass m includes everything except the two balls.

(i.e. unrolled) portion of the rug is at rest relative to the floor. Examples of analogous inverse rocket problems (in which an object is *accreting* elements of mass that are initially at rest, rather than *ejecting* elements of mass that end up at rest) include a raindrop falling through stationary mist particles that agglomerate onto the drop [18, 19], a snowball rolling down a slope covered with snow that sticks onto the ball [20], or a grappling hook launched into the air and pulling rope off a stationary coil [21]. In all four of these forward and inverse rocket examples, the frame S is chosen to be the earth. If one instead jumped into an inertial frame moving at constant velocity relative to the earth [22] then $\vec{u}$ would no longer be zero and so $\vec{F}_{\rm ext}$ would no longer be equal to $\mathrm{d}\vec{p}/\mathrm{d}t$; nevertheless, equation (2) through (4) would continue to give correct expressions for the motion.

3. Discrete mass ejection

Now suppose the ejecta is emitted in the form of finite chunks of mass separated by discrete intervals of time (rather than infinitesimal element that are continuously emitted). Appendix B provides the key equation used to find the velocity of such a rocket after it ejects a chunk of total mass $\Delta m > 0$. Consider an artificial but simple example chosen to illustrate the concept. As illustrated in figure 1, two balls of mass $0.5\,\Delta m$ initially compress two ideal springs S of stiffness constant k by an amount x. The balls are then released at exactly the same instant. Two ideal ropes R cause them to swing around through a quarter circle until both ropes simultaneously run into blades B that cut them, releasing the balls leftward. Applying conservation of mechanical energy to the spring release and conservation of linear momentum to the rope rotation and cutting, the payload of mass m will get propelled rightward at speed

$$v = x\sqrt{\frac{2k\Delta m}{(m+\Delta m)\,m}} \tag{6}$$

in the absence of external forces. The source of the final kinetic energy gained by the rocket and ejecta is the initial potential energy stored in the compressed springs. In contrast, the source of the thrust is the tension in the ropes. The horizontal component of the tension in each rope pulls on its attached ball to drive the balls leftward, and it pulls on the pivot P to drive the payload rightward. The same idea would explain two identical balls rotating about the midpoint of an ideal string connecting them (as discussed in more detail in appendix B). If the string is cut, one ball (called the ejecta) flies off in one direction and the other ball (designated as the payload) flies off with equal speed in the other direction. The propulsion energy comes from the mechanism used to initially spin up the system, as in the case of the centrifugal accelerator proposed by SpinLaunch [23] whereby a payload (of up to 200 kg mass) is attached to a rotating arm and its speed increased inside an evacuated drum from zero to about 2 km s^{-1} before it is released upward from earth's surface. Similar to this proposal[2], a rotating

[2] If the SpinLaunch accelerator propels a vehicle upward into the atmosphere, then the accelerator must recoil downward [44], but if the launcher is rigidly attached to the earth, the recoil impulse will be too weak to impart any noticeable velocity to it.

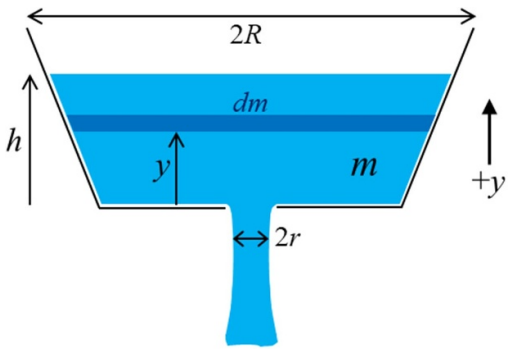


Figure 2. Water leaking out of a hole in the bottom of a cylindrically symmetric tank. The container is assumed to be of negligible mass compared to the water.

ring-shaped space station (whose center is initially stationary) that ejects a module tangentially off of its periphery would be propelled in the opposite direction [24].

A more conventional example of a discretely emitting rocket is the put rocket or trolley car [25, 26] propelled by throwing stones backward from its tail end. A delightful challenge problem worth discussing with students is why a *different* final velocity of these vehicles is obtained if two such stones are thrown simultaneously rather than sequentially one after the other [11, 27]. To resolve the ambiguity about whether $\vec{v}$ in equation (4) should be the *initial* velocity $\vec{v}_\mathrm{i}$ or the *final* velocity $\vec{v}_\mathrm{f}$ of the rocket (as briefly discussed in appendix B), suppose the ejection process converts a given amount of initial potential energy of some compressed springs into final kinetic energy of the ejecta and rocket, as in the discussion of figure 1 above. In that case, simultaneous ejection of two identical stones results in a larger final speed of the rocket than does sequential ejection of them [28].

4. Defining the velocity of the rocket

What exactly is meant by $\vec{v}$, given that different parts of any realistic rocket have different velocities? (For example, astronauts aboard it can move around, and there are likely to be motorized objects and moving fluids on the rocket.) Despite what one might guess, it does *not* in general refer to the velocity of the *center of mass* of the rocket, but instead to the *average* velocity of the rocket.

An example makes the distinction between these two velocities clear. Consider a cylindrically symmetric tank of water of upper radius R that is open to the atmosphere along its top face and that has a circular hole centered on its bottom face, as sketched in figure 2. Denoting a time derivative by an overdot, the y-component of the average velocity of the water inside the tank is [29]

$$v_y = \frac{1}{m}\int \dot{y}\,\mathrm{d}m = \frac{\int_0^h \dot{y}A(y)\,\mathrm{d}y}{\int_0^h A(y)\,\mathrm{d}y} \tag{7}$$

where $\mathrm{d}m = \rho A(y)\mathrm{d}y$ is the mass of a horizontal layer of water of (incompressible) density ρ and infinitesimal thickness $\mathrm{d}y$ at height y above the bottom of the tank. The internal cross-sectional area of the tank is $A(y)$ at this height, and the y-component of the velocity of the water in the layer is $\dot{y} < 0$. The surface of the water is at (time-varying) height h above the bottom of the tank.

For example, for a right cylindrical tank (having a constant internal radius R) all of the water inside it is moving at the same downward velocity (by continuity) which equals the average velocity [30]. Therefore $v_y = \dot{h}$. In contrast, the center of mass of the water inside the tank is at height $y_\mathrm{cm} = h/2$ and thus $v_\mathrm{cm} = v/2$. If the tank is funnel-shaped [31], there is a different relation between v_cm and v, but they remain unequal to each other[3].

At first glance, it seems puzzling that these two velocities are different. Equation (7) certainly looks like it accords with the usual textbook definition of center-of-mass velocity. In fact, the issue is that the

[3] This funnel example also makes it clear that the velocity $\vec{v}$ used in equation (4) is not in general equal to the velocity the ejecta had just prior to its emission. In other words, $\vec{v}_\mathrm{rel}$ is *not* equal to the velocity of the water just above the hole subtracted from the velocity of the water just below the hole in figure 2.

system here has *variable mass*. Explicitly, the center-of-mass height is

$$y_{\rm cm} = \frac{1}{m}\int y\,{\rm d}m \quad \Rightarrow \quad \int y\,{\rm d}m = my_{\rm cm}. \tag{8}$$

Differentiate both sides of this second equation with respect to time to obtain

$$mv_y = mv_{{\rm cm}\,y} + \dot{m}y_{\rm cm} \tag{9}$$

using equation (7). For example, equation (9) can be checked in the case of a right cylindrical tank as

$$mv_y = m\frac{v_y}{2} + \rho A\dot{h}\frac{h}{2} \tag{10}$$

using $v_{\rm cm} = v/2$, $y_{\rm cm} = h/2$, and $m = \rho Ah$ for constant A. Substituting $\dot{h} = v_y$ verifies that the right-hand side of equation (10) correctly simplifies to mv_y. The point is that, although the center-of-mass height is defined as the mass-weighted average height of water in the tank, the center-of-mass velocity is *not* defined as the mass-weighted average velocity of water in the tank. Instead, the center-of-mass velocity is defined as the time derivative of the center-of-mass height. Equation (9) shows that the center-of-mass velocity $\vec{v}_{\rm cm}$ is not equal to the average velocity $\vec{v}$ for an object whose mass varies with time.

It is instructive to explicitly calculate the thrust in the case of water flowing out of a small hole in a right cylindrical tank. A small hole is one such that the (minimum) cross-sectional area πr^2 of the jet of water flowing out of it[4] is much less than the internal cross-sectional area πR^2 of the tank. Substituting equation (4) into (3) and taking the y-component results in

$$F_{{\rm thrust}\,y} = \left(u_y - v_y\right)\frac{{\rm d}m}{{\rm d}t}. \tag{11}$$

The mass of the water in the tank is $m = \rho\pi R^2 h$ so that

$$\frac{{\rm d}m}{{\rm d}t} = \rho\pi R^2 v_{y.}. \tag{12}$$

Continuity implies that the flow rate of water inside the tank equals that in the jet of water issuing out of the hole,

$$\pi R^2 v_y = \pi r^2 u_y. \tag{13}$$

Substitute equation (12) into (11), and then use (13) to replace both instances of v_y to get

$$F_{{\rm thrust}\,y} = u_y\left(1 - \frac{r^2}{R^2}\right)\rho\pi r^2 u_y \approx \rho\pi r^2 u_y^2. \tag{14}$$

The approximation follows from the fact that $r^2 << R^2$, which also implies that one can use Torricelli's law [32] in the form

$$u_y = -\sqrt{2gh} \tag{15}$$

for the downward velocity of the water out of the hole. Substituting equation (15) into (14) gives the final expression for the thrust in terms of measurable quantities,

$$F_{{\rm thrust}\,y} = 2\rho\pi r^2 gh. \tag{16}$$

An interesting conceptual discussion about why the numerical prefactor on the right-hand side of this equation is 2 instead of 0 or 1 can be found in Marchewka and Navot [33], along with a confirming experimental validation of it. A similar experiment uses an Atwood machine where one of the fixed masses is replaced with a funnel that is discharging water [34]. Keep in mind, however, that the thrust does not act solely on the nozzle of the funnel [35] but instead on the entire funnel and water it instantaneously contains.

[4] The minimum cross-sectional area of the jet is smaller than the area of the hole from which it issues by the factor of the coefficient of discharge, an effect known as the *vena contracta* [45, 46].

5. Conclusion

This article discusses the definition of thrust for a rocket whose velocity increases in some direction because it is ejecting mass in the opposite direction. The mass can either be emitted continuously as a flow of infinitesimal elements $\mathrm{d}m$ (in which case the thrust on the rocket is described as a force) or discontinuously in the form of discrete chunks Δm (such that the thrust should instead be analyzed in terms of the impulses the ejecta exert on the rocket). The mathematical definitions given in equation (3) or (B9) spell out two conditions that must be true if the thrust on a rocket is to be nonzero. First, its mass must decrease. For example, a flying squid propels itself through the air by ejecting water from its mantle [36] like a bottle rocket. The second condition is that the ejecta must be emitted at a velocity different than that of the rocket. In other words, there must be a nonzero relative velocity between them, which is defined to be the exhaust velocity in equation (4). A case was debated in the literature [15, 16] of sand falling from a funnel into a moving cart. An investigator held the funnel in his hand a constant distance above the cart but translated his hand horizontally so that it was always directly above the cart (for the purpose of ensuring that the sand landed in the cart, rather than before or behind it). The sand exerts zero thrust on the cart in this situation because the sand has the same horizontal velocity just before and after it lands in the cart, so that $\vec{v}_{\rm rel}=0$.

A subtly different definition of thrust applies to calculating the propulsive force of a propeller on a helicopter or submarine. In that case $\mathrm{d}m/\mathrm{d}t$ refers to the rate at which air or water flows through the area swept out by the rotating blades. Fluid mass is being pushed away from the craft, but the vehicle is not losing mass in the process. Nevertheless, an equation of the same form as (3) results, where the exhaust velocity is now the change in the speed of the fluid due to the action of the propeller [37, 38].

A common source of confusion about rocket motion is the question of under what circumstances one can describe the time derivative of the linear momentum of the rocket as being equal to the net external force on it? The answer is: If and only if the velocity $\vec{v}$ of the ejected mass is zero in the same inertial frame of reference that the momentum $\vec{p}=m\vec{v}$ of the rocket is being measured. In practice, such a situation is more commonly encountered for the inverse rocket problem of an object accreting (rather than ejecting) mass, such as a raindrop falling through stationary mist and gaining mass as some of that mist agglomerates onto the drop. In discussing these special cases of rocket problems with students, it is worth emphasizing the distinction between the time derivative of the momentum of the *rocket alone* and that of the *combined system of the rocket and all of its emitted ejecta*. The latter combined system has constant mass m_0 and thus the acceleration of its center of mass *always* equals the net external force $\vec{F}_{\rm ext}$ acting on it divided by m_0. In contrast, the acceleration of a rocket of variable mass m is not in general equal to the net external force acting on it divided by m.

This article also considers the distinction between the *average* velocity and the *center-of-mass* velocity of a rocket of variable mass m. Given that the calculation of thrust is directly proportional to the difference between the velocity $\vec{u}$ of the ejecta and the velocity $\vec{v}$ of the rocket, it is important to understand how these two rocket velocities differ and why equation (7) is the correct formula to use for $\vec{v}$ in calculating (4) in that water tank example.

Acknowledgment

Thanks go to Yiftah Navot for pointing out the difference between the average and center-of-mass velocities for a variable-mass system, in contrast to the terminology used in reference 31 (although the calculations in it remain correct). The views expressed in this article are those of the author and do not reflect the official policy or position of the US Naval Academy, the Department of the Navy, the Department of War, or the US Government.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

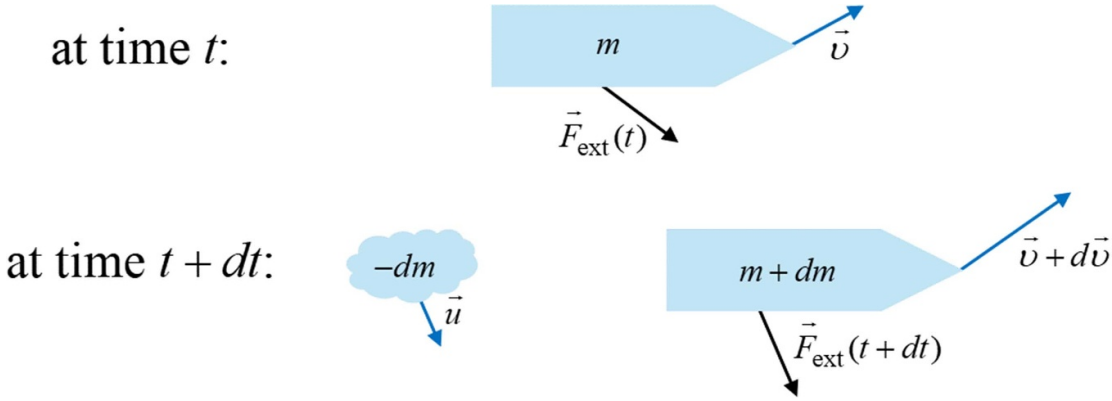


Figure A1. Initial and final masses, velocities, and net external force on a rocket.

Appendix A: Derivation of equation (2)

Consider figure A1 which sketches a continuously emitting rocket at an arbitrary time t and at an infinitesimal time $\mathrm{d}t$ later, after it has ejected an infinitesimal element of mass $-\mathrm{d}m > 0$. The five arrows indicating the velocities $\vec{v}$ and $\vec{v}+\mathrm{d}\vec{v}$ of the rocket, the velocity $\vec{u}$ of the ejected element of mass, and the initial and final net external force $\vec{F}_{\text{ext}}$ on the rocket are drawn in arbitrary directions, although in practice they are likely to point rightward or leftward. The total linear momenta at times t and $t+\mathrm{d}t$ are respectively

$$\vec{P} = m\vec{v} \text{ and } \vec{P}+d\vec{P} = (m+\mathrm{d}m)(\vec{v}+\mathrm{d}\vec{v}) + (-\mathrm{d}m)(\vec{u}). \tag{A1}$$

Since the total mass is m both initially and finally, and is thus constant during this ejection process, Newton's second law can be safely written as

$$\vec{F}_{\text{ext}} = \frac{\mathrm{d}\vec{P}}{\mathrm{d}t}, \tag{A2}$$

where the external force on the infinitesimal ejected mass element must be infinitesimal (according to Newton's second law) and can thus be neglected compared to the external force on the rocket. Equation (A2) can be rewritten as

$$\vec{F}_{\text{ext}} = \frac{\left(\vec{P}+\mathrm{d}\vec{P}\right)-\left(\vec{P}\right)}{dt} = \frac{m\,\mathrm{d}\vec{v}+\mathrm{d}m\,\vec{v}-\mathrm{d}m\,\vec{u}}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} - (\vec{u}-\vec{v})\frac{\mathrm{d}m}{\mathrm{d}t} \tag{A3}$$

after dropping the second-order-small term $\mathrm{d}m\,\mathrm{d}\vec{v}$ in the numerator. Substituting the definitions of thrust and exhaust velocity from equations (3) and (4) gives (2) as desired. Equation (2) applies generally to variable-mass problems and not just to traditional rockets. For example, it can be applied to the pulling of paper towels off a roll [39], as well as to falling chain problems [40, 41].

Specifically, define the $+x$ axis to be the direction of motion of the rocket at the current instant. The x component of equation (2) is

$$F_{\text{ext }x} + F_{\text{thrust }x} = ma_x \tag{A4}$$

where the acceleration of the rocket is $\vec{a} = \mathrm{d}\vec{v}/\mathrm{d}t$. If mass is ejected backward relative to the rocket, then $u_x < v_x \Rightarrow v_{\text{rel }x} < 0$. Given that $\mathrm{d}m/\mathrm{d}t$ is also negative (as the rocket loses mass), equation (3) then implies that $F_{\text{thrust }x} > 0$ as expected: the thrust provides a forward acceleration of the rocket, which opposes gravity and air drag for a rocket rising vertically upward from the surface of the earth for example.

Appendix B: Derivation of the rocket velocity due to discrete mass ejection

Figure B1 sketches a rocket of initial mass m and velocity $\vec{v}_{\text{i}}$ which ejects a noninfinitesimal element of mass $\Delta m > 0$ at velocity $\vec{u}$. Consequently the rocket has a final mass $m-\Delta m$ and velocity $\vec{v}_{\text{f}}$. An external (time-varying) force $\vec{F}_{\text{ext}}$ acts on the system of rocket and ejecta. The linear momenta of the system

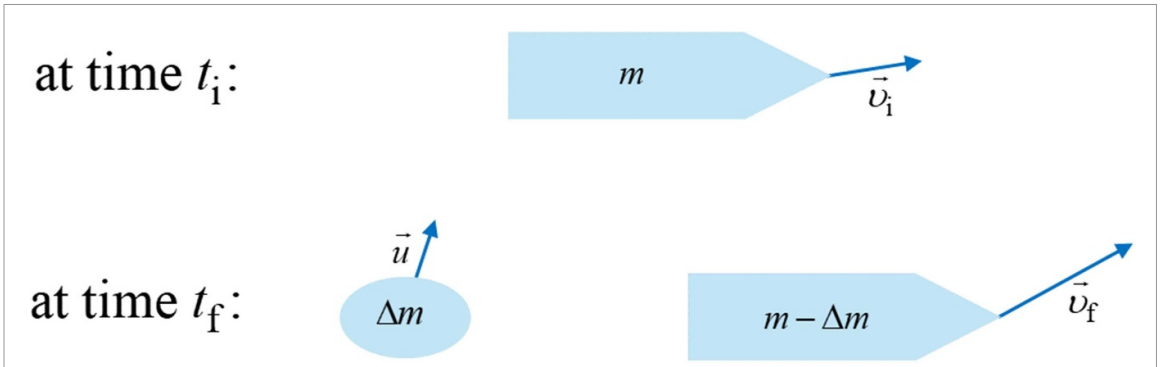


Figure B1. A rocket that suddenly ejects a chunk of mass Δm.

at times t_i and t_f are respectively

$$\vec{P}_i = m\,\vec{v}_i \text{ and } \vec{P}_f = (m-\Delta m)\,(\vec{v}_f) + (\Delta m)\,(\vec{u}) \tag{B1}$$

so that $\vec{P}_f - \vec{P}_i$ becomes

$$\Delta\vec{P} = (m-\Delta m)\,(\vec{v}_f - \vec{v}_i) + (\Delta m)\,(\vec{u} - \vec{v}_i)\,. \tag{B2}$$

This change in the system momentum equals the impulse $\vec{I}$ due to the net external force acting on the system between the initial and final times,

$$\Delta\vec{P} = \int_{t_i}^{t_f} \vec{F}_{\text{ext}}\,\mathrm{d}t = \vec{I}_{\text{ext on rocket}} + \vec{I}_{\text{ext on ejecta}}\,. \tag{B3}$$

Likewise, the impulse on the ejecta alone is equal to its change in linear momentum, where the ejecta is initially part of the rocket traveling at velocity $\vec{v}_i$ but finally is moving independently through space at velocity $\vec{u}$,

$$(\Delta m)\,(\vec{u} - \vec{v}_i) = \vec{I}_{\text{ext on ejecta}} + \vec{I}_{\text{rocket on ejecta}}\,. \tag{B4}$$

Substitute equations (B3) and (B4) into (B2) to get

$$\vec{I}_{\text{ext on rocket}} + \vec{I}_{\text{ext on ejecta}} = (m-\Delta m)\,(\vec{v}_f - \vec{v}_i) + \vec{I}_{\text{ext on ejecta}} + \vec{I}_{\text{rocket on ejecta}}\,. \tag{B5}$$

In accordance with Newton's third law, the impulse due to the ejecta on the rocket is equal and opposite to the impulse due to the rocket on the ejecta. Define the former as the 'thrust impulse' on the rocket,

$$\vec{I}_{\text{thrust}} \equiv \vec{I}_{\text{ejecta on rocket}} = -\vec{I}_{\text{rocket on ejecta}}\,. \tag{B6}$$

Also, define the 'external impulse' to be that on the rocket alone,

$$\vec{I}_{\text{ext}} \equiv \vec{I}_{\text{ext on rocket}}\,. \tag{B7}$$

Using the two impulses defined in equations (B6) and (B7), equation (B5) can be rewritten as

$$\vec{I}_{\text{ext}} + \vec{I}_{\text{thrust}} = m_f\Delta\vec{v} \tag{B8}$$

where $m_f \equiv m - \Delta m$ is the final mass of the rocket and $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$ is its change in velocity. Equation (B8) is the analog of equation (2) and can be used to calculate $\vec{v}_f$ (by dividing the sum of the two impulses by m_f and then adding $\vec{v}_i$ to that result).

In practice, the thrust impulse is calculated as

$$\vec{I}_{\text{thrust}} \approx (\Delta m)\,(\vec{v}_i - \vec{u}) \tag{B9}$$

i.e. by approximating the impulse on the ejecta in equation (B4) as being solely due to its interaction force with the rocket (not including any external forces that act on the ejecta). This approximation is reasonable if the rocket-ejecta interaction force is much larger in magnitude than the net external force

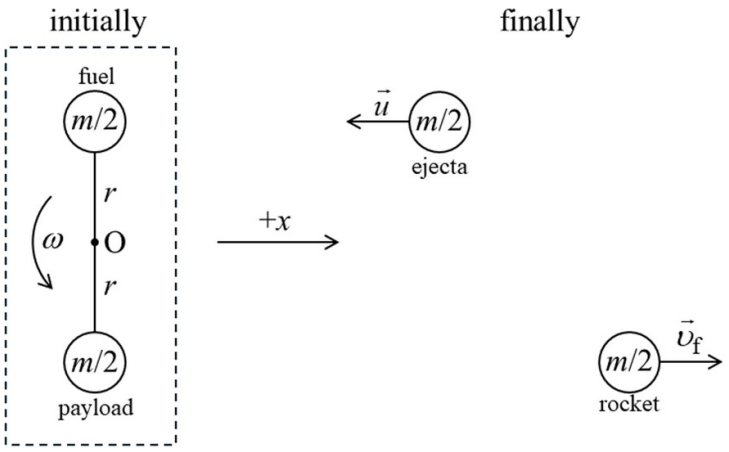


Figure B2. A rocket that initially carries both a payload and a fuel cartridge, with total mass m inside the dashed box whose average velocity is zero. Finally, it has ejected the fuel of mass $m/2$ leftward at velocity $\vec{u}$, and the remaining rocket of mass $m/2$ travels rightward at velocity $\vec{v}_{\rm f}$ (defining the direction of the $+x$ axis).

on the ejecta, and if $t_{\rm i}$ and $t_{\rm f}$ are both chosen to lie close enough in time to the instant of ejection such that any external forces do not have sufficient time to significantly change the momentum of the ejecta.

The similarity between equations (2) and (B8) is strengthened by noting that m on the right-hand side of equation (2) is more exactly equal to $m+{\rm d}m$ (and is thus analogous to $m-\Delta m$) if the second-order-small term ${\rm d}m\,{\rm d}\vec{v}$ in equation (A3) were not dropped. To put it differently, if $\Delta m << m$ then one can approximate $m_{\rm f}$ in equation (B8) by the initial rocket mass m. Alternatively, one could define the thrust impulse to be relative to the *final* (rather than the *initial*) velocity of the rocket [42] by replacing $\vec{v}_{\rm i}$ by $\vec{v}_{\rm f}$ in equation (B9), in which case $m_{\rm f}$ would get replaced by the initial rocket mass m in equation (B8), as the reader is encouraged to verify.

Define the $+x$ axis to be the direction of motion of the rocket. The x component of equation (B8) is

$$I_{{\rm ext}\ x}+I_{{\rm thrust}\ x}=(m_{\rm f})\,(v_{\rm fx}-v_{\rm ix})\,. \tag{B10}$$

If mass is ejected backward relative to the rocket, then $u_x<v_{\rm ix}$. Given that $\Delta m>0$, equation (B9) then implies that $I_{{\rm thrust}\ x}>0$ so that the thrust acts to push a discretely emitting rocket forward, just as was the case for the continuously emitting rocket discussed at the end of appendix A.

As a concrete example, consider the simple system sketched in figure B2. Two balls in deep space, each of mass $m/2$, are initially rotating counter-clockwise at angular speed ω about their center point O that is at rest (relative to some inertial frame S). Each ball is connected to O by an ideal string of length r. The string is suddenly cut, causing one ball (treated as the ejected fuel) to fly off leftward and the other ball (containing a device or instrument of interest so that it can be considered to be the rocket payload) to fly off rightward. Equation (B10) with approximation (B9) takes on the form

$$(\Delta m)\,(v_{\rm ix}-u_x)=(m_{\rm f})\,(v_{\rm fx}-v_{\rm ix}) \tag{B11}$$

since there is no external force in deep space. The ejecta has mass $\Delta m=m/2$ and x-component of its velocity $u_x=-r\omega$. The rocket has initial and final masses of $m_{\rm i}=m$ and $m_{\rm f}=m/2$, and the x-components of its initial and final average velocities are $v_{\rm ix}=0$ and $v_{\rm fx}=r\omega$, respectively. Thus equation (B11) becomes

$$(m/2)\,(0+r\omega)=(m/2)\,(r\omega-0) \tag{B12}$$

which is satisfied by inspection. The left-hand side of this equation equals the nonzero thrust impulse that propels the payload rightward after ejection. In contrast, prior to ejection the payload spends equal time intervals traveling rightward as leftward, and thus its velocity is zero *on average*.

ORCID iD

Carl E Mungan 0000-0001-7084-5402

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