The Dual-Spring-Cannon-Propelled Cart

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Abstract
Two identical spring-loaded cannons pointing backward are mounted on a frictionless railcart. The cannons are to be fired to propel the cart forward as rapidly as possible. Does it matter whether the two cannonballs are launched simultaneously or sequentially? The velocities and kinetic energies of the two balls and of the cart in various reference frames are analyzed in detail in order to explain the answer.

1. Introduction
A dual-cannon “rocket” cart is sketched in Fig. 1. A similarly powered trolley has been considered by Brun [1] with the assumption that each ball has the same final velocity relative to the initial velocity of the cart, regardless of when it is launched. Pinheiro [2], on the other hand, assumes that a ball’s final velocity is to be measured relative to the final velocity of the cart. Either way, the assumption that the relative exhaust speed of the balls will always be the same is questionable, at least for nonchemically powered cannons [3] such as the spring-loaded ones treated here. Edmonds [4] has specifically shown that the cart ends up with a different final velocity if the two balls are launched simultaneously than if they are launched one after the other. He first assumes that each ball has the same exhaust speed relative to the final velocity of the cart regardless of when it is launched. Next he considers the possibility that the total kinetic energy gained by the two balls and the cart is the same, whether the balls are released sequentially or simultaneously, but he does not spell out how that can be accomplished. In fact, in a follow-up letter [5] he says that the exhaust speeds must be carefully calculated and the launch mechanism...
adjusted (in some unspecified fashion) to ensure it happens. His follow-up was in part a response to comments by Scaife [6] and by Hinson [7]. Hinson showed that the cart will end up with the same final velocity when the two balls are released simultaneously as when they are launched sequentially in the limit that the mass of the balls is much less than that of the cart, \( m \ll M \). The standard rocket equation is obtained in that limit when a continuous stream of exhaust is emitted, such as in the form of burning gas [8]. Specifically, during an infinitesimal emission time \( dt \) an infinitesimal amount of mass \( dm \) is launched out the tail, rather than the impulsive non-infinitesimal amount \( m \) in the present problem. Scaife has further commented that it is more efficient (in the sense of leading to a higher final rocket velocity for a fixed initial amount of fuel) to emit the gas in chunks (or clouds) for which all internal portions have the same exhaust velocity (and which thereby are dynamically equivalent to discrete balls). That claim anticipates Gowdy’s “perfect” rocket [3] and also verifies Edmonds’s hypothesis in his second letter [5] that for sequential launching the second ball should have the same kinetic energy (and hence velocity) as the first ball, in order to maximize the final velocity of the rocket.

In the present article, the situation is reconsidered, but this time with a specific launch mechanism (namely spring-loaded cannons). In that way, no ad hoc assumption about the exhaust speeds of the balls is necessary. Instead, an explicit calculation of those velocities is carried out, by combining the conservation laws of mechanical energy and of linear momentum for the isolated system of balls plus cart. The result is a well-defined problem (free of questionable assertions), that is solvable at the introductory physics level (using only elementary classical mechanics), and with three possible answers (namely simultaneous firing leads to the greatest final cart speed, or sequential firing does, or it is a tie) which all appear plausible and thus find proponents when posed as a ConcepTest clicker question [9] to students.

Prior to revealing the correct answer, anticipation will be heightened if a class discussion includes consideration of the following two points [10, 11]. On the one hand, the gain in kinetic energy of the system of two cannonballs and cart in Fig. 1 must be equal to the elastic potential energy \( 2E \) lost by the two cannons. This equality holds in any inertial frame of reference, which might suggest that the order of firing is irrelevant. On the other hand, consider the center-of-mass reference frame of the ground (in which the cart and balls are initially at rest). If the two cannons are fired sequentially, then the second ball will have an exhaust velocity relative to an initially forward moving cart. Its velocity relative to the ground will then normally be smaller than that relative velocity. Hence, if the two cannonballs are fired with nearly (or exactly) the same relative exhaust velocities, the second ball will acquire less kinetic energy relative to the ground than the first ball. That leaves more kinetic energy to be gained by the cart, which is similar to the Oberth effect for rocket propulsion [12]. This idea suggests that sequential firing might have an advantage over simultaneous firing.

2. Analysis using the nonrelativistic conservation laws of classical mechanics

This problem is an excellent example of a situation where a straightforward mathematical analysis is more convincing than conceptual arguments. Begin by considering the general problem of a cart of mass \( M \) carrying a single cannonball of mass \( m \). (Let \( M \) be the mass of the cart without the ball.) Suppose the cart is initially traveling forward at speed \( U \) before it fires the ball. The cannon is taken to be ideal in the sense that 100% of its elastic
potential energy $E$ gets converted into translational kinetic energy of the cart and ball. Adopt a reference frame moving with the initial speed $U$ of the cart. After firing the cannon, denote the final speed of the ball and of the cart in this frame as $\nu$ and $V$, respectively. Conservation of linear momentum implies that

$$mv = MV$$

(1)

while conservation of mechanical energy leads to

$$E = \frac{1}{2} mv^2 + \frac{1}{2} MV^2 .$$

(2)

Simultaneous solution of Eqs. (1) and (2) gives

$$\nu = \sqrt{\frac{2EM}{(M+m)m}}$$

and

$$V = \sqrt{\frac{2Em}{(M + m)M}} .$$

(3)

Armed with this result, the final cart speed can be determined when it initially carries two balls as in Fig. 1. First, if the two cannons are fired simultaneously with the cart initially at rest relative to the ground, then replace $m \rightarrow 2m$ and $E \rightarrow 2E$ in the expression for $V$ to get

$$V_{\text{simultaneous}} = \sqrt{\frac{8Em}{(M + 2m)M}} .$$

(4)

Next, for the sequential firing case, after the first cannon has been fired, the final speed $V$ of the cart becomes the initial speed $U$ for the second firing, except that it is necessary to replace $M \rightarrow M + m$ because one cannonball is still left onboard. Thus, the speed of the cart after the first cannon firing but before the second firing is

$$U = \sqrt{\frac{2Em}{(M + 2m)(M + m)}} .$$

(5)

After the second firing, the cart has a speed given by precisely $V$ in Eq. (3) relative to speed $U$. Thus the final speed of the cart relative to the ground becomes

$$V_{\text{sequential}} = V + U = \sqrt{\frac{2Em}{(M + m)M}} + \sqrt{\frac{2Em}{(M + 2m)(M + m)}} .$$

(6)

The ratio of Eq. (6) to (4) is

$$\frac{V_{\text{sequential}}}{V_{\text{simultaneous}}} = \frac{\sqrt{M + \sqrt{M + 2m}}}{2\sqrt{M + m}} .$$

(7)

This expression reduces to unity if $m \ll M$, so that one gets a tie in that limit, as Hinson [7] remarked. However, it is less than 1 otherwise. For example, it is equal to 96.6% if $m = M$, which admittedly requires heavy cannonballs. (Interestingly, the ratio is independent of $E$, however, and so it does not matter how fast the balls are launched.)

Therefore the two cannons should be fired simultaneously rather than sequentially, in order to maximize the final speed of the cart. To better understand this result, consider the speeds and kinetic energies of the launched cannonballs. In the simultaneous case, the speed and kinetic energy of the two balls relative to the ground (or equivalently, relative to the initial velocity of the cart) are respectively

$$\nu_{\text{simultaneous}} = \sqrt{\frac{2EM}{(M + 2m)m}}$$

and

$$K_{\text{balls\simultaneous}} = \frac{1}{2} (2m)\nu_{\text{simultaneous}}^2 = \frac{2EM}{M + 2m} .$$

(8)

where $\nu_{\text{simultaneous}}$ was obtained by again replacing $m \rightarrow 2m$ and $E \rightarrow 2E$ in the expression for $\nu$ in Eq. (3), just as was done to get Eq. (4). As a check, the kinetic energy of the cart relative to the ground is
\[ K_{\text{cart\ simultaneous}} = \frac{1}{2} M V_{\text{simultaneous}}^2 = \frac{4Em}{M + 2m} \]  

(9)

so that one correctly obtains

\[ K_{\text{balls\ simultaneous}} + K_{\text{cart\ simultaneous}} = 2E . \]  

(10)

Alternatively, the speed of the simultaneously fired cannonballs relative to the final velocity of the cart is

\[ v'_{\text{simultaneous}} = v_{\text{simultaneous}} + V_{\text{simultaneous}} = \sqrt{\frac{2E(M + 2m)}{Mm}} \]  

(11)

with corresponding kinetic energy of

\[ K_{\text{balls\ simultaneous}} = \frac{1}{2}(2m)v'_{\text{simultaneous}}^2 = \frac{2E(M + 2m)}{M} . \]  

(12)

The cart has no final kinetic energy in this frame of reference. However, the system is initially moving backward at speed \( v_{\text{sequential}} \) in this frame, so that the initial mechanical energy of the system is

\[ 2E + \frac{1}{2}(M + 2m)V_{\text{simultaneous}}^2 = \frac{2E(M + 2m)}{M} \]  

(13)

in agreement with Eq. (12). This result verifies the first point suggested for class discussion at the end of Sec. 1. It demonstrates that, unlike kinetic energy, potential energy is frame independent [13]. Each cannon has initial elastic energy \( E \) stored in its compressed spring regardless of the frame of reference adopted, whereas the kinetic energies of the balls and cart have different values depending on the motion of the frame.

In the sequential case, the speed and kinetic energy of the first ball relative to the ground are respectively

\[ v_{\text{first}} = \sqrt{\frac{2E(M + m)}{(M + 2m)m}} \]  

(14)

\[ K_{\text{first ball\ sequential}} = \frac{1}{2} mv^2_{\text{first}} = \frac{E(M + m)}{M + 2m} \]

where \( v_{\text{first}} \) was obtained by replacing \( M \rightarrow M + m \) in the expression for \( v \) in Eq. (3), just as was done to get Eq. (5). In analogy to Eq. (6), the speed and kinetic energy of the second ball relative to the ground are respectively

\[ v_{\text{second}} = |v - U| = \sqrt{\frac{2EM}{(M + m)m}} - \sqrt{\frac{2Em}{(M + 2m)(M + m)}} \]

(15)

and \( K_{\text{second ball\ sequential}} = \frac{1}{2} mv^2_{\text{second}} \).

As discussed at the end of Sec. 1, the second ball’s backward ground velocity is diminished by the forward velocity of the cart; the absolute value bars are needed because either square root term can be larger than the other, depending on the relative sizes of \( m \) and \( M \). For example, \( v = U \) when \( m/M = 1 + \sqrt{2} \) in which case the second ball has a final speed of zero relative to the ground! In general, since the kinetic energy of the cart relative to the ground is

\[ K_{\text{cart\ sequential}} = \frac{1}{2} MV^2_{\text{sequential}} \]  

(16)

where \( V_{\text{sequential}} \) is given by Eq. (6), one again correctly finds that

\[ K_{\text{first ball\ sequential}} + K_{\text{second ball\ sequential}} + K_{\text{cart\ sequential}} = 2E \]  

(17)

after some algebra. Although \( v_{\text{second}} \) is smaller than \( v_{\text{simultaneous}} \) when \( m/M < 3 + 2\sqrt{3} \), \( v_{\text{first}} \) is sufficiently larger than \( v_{\text{simultaneous}} \) that \( K_{\text{cart\ sequential}} \) always ends up being smaller than \( K_{\text{simultaneous}} \) in contrast to the suggestion implied by the Oberth effect above.
3. Closing comments

One can use these results to show that the exhaust speeds of the cannonballs (relative to either the initial or final velocities of the cart at the instant of their firings) are all different from each other, as listed in Table 1. Only in the limit $m \ll M$ do they all converge to the same value of $\sqrt{2E \over m}$. Otherwise, the exhaust speeds relative to the initial cart velocity are in the order $v'_{\text{first}} > v''_{\text{second}} > v_{\text{simultaneous}}$ while those relative to the final cart velocity are in the opposite order $v_{\text{simultaneous}} > v'_{\text{second}} > v'_{\text{first}}$. A simple example of these inequalities can be verified by putting $m = M$ into the six formulas in Table 1.

The solution that simultaneous ejection gives a larger final railcart speed than sequential ejection of two balls becomes more intuitive in the opposite limit where the final payload is much lighter than the exhausted fuel. In the sequential case, after the first half-mass $m$ is ejected, both it and the second half-mass $m$ (plus its tiny attached $M$) each gain $E/2$ amount of kinetic energy by symmetry; then after the second spring is fired, there is almost no change in the kinetic energy of the second (now detached) half-mass because the payload $M$ is so light compared to it. (The speed of a moving car does not noticeably change if the driver flicks a cigarette butt out the window!) Thus each fuel half-mass $m$ ends up with $E/2$ amount of kinetic energy, and so the cart must end up with the remaining $E$ amount of released kinetic energy (all relative to the ground). On the other hand, in the simultaneous case, almost all $2E$ goes into the payload and little into the heavy full-fuel-half $2m$.

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**Table 1.** Speeds of the balls relative to either the initial or final velocities of the cart at the instant of firing. Here $v_{\text{simultaneous}}$ is from Eq. (8), $v'_{\text{simultaneous}}$ is from Eq. (11), $v_{\text{first}}$ is from Eq. (14), $v'_{\text{first}} = v_{\text{first}} + U$ from Eqs. (5) and (14), $v''_{\text{second}} = v$ from Eq. (3), and $v'_{\text{second}} = v + V$ from Eq. (3).

<table>
<thead>
<tr>
<th>ball</th>
<th>exhaust speed relative to initial cart speed</th>
<th>exhaust speed relative to final cart speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>simultaneous</td>
<td>$v_{\text{simultaneous}} = \sqrt{2EM \over (M + 2m)m}$</td>
<td>$v'_{\text{simultaneous}} = \sqrt{2E(M + 2m) \over Mm}$</td>
</tr>
<tr>
<td>first sequential</td>
<td>$v_{\text{first}} = \sqrt{2E(M + m) \over (M + 2m)m}$</td>
<td>$v'_{\text{first}} = \sqrt{2E(M + 2m) \over (M + m)m}$</td>
</tr>
<tr>
<td>second sequential</td>
<td>$v''_{\text{second}} = \sqrt{2EM \over (M + m)m}$</td>
<td>$v'_{\text{second}} = \sqrt{2E(M + m) \over Mm}$</td>
</tr>
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ejected as a unit. The ratio of final payload kinetic energies in the two cases is therefore
\[
\frac{1}{2} M V^2_{\text{sequential}} = \frac{E}{2E} = \frac{1}{2} \quad \text{(18)}
\]
which agrees with Eq. (7) evaluated in the limit \( m \gg M \).

In conclusion, the efficiency or utilisation [14] of the fuel to accelerate the payload is maximized by launching all of the fuel out the tail end at once, rather than dribbling it out in sequential bits. However, that is only true if the rocket can maintain the same launch energy per unit mass of fuel either way! Specifically, to ensure that is true in the present setup, two cannons are used to fire the balls simultaneously. The rocket would be lighter (and less costly to build) if it instead had only one cannon which was used to fire the two balls sequentially. For a real rocket, there are trade-offs in the weight and power of the engine that favor sequential operation.

Finally, it would be interesting to explore what happens if the second spring cannon were fired after the first cannon is fired but before the first spring has fully decompressed and released its ball. Presumably the results would smoothly interpolate between the simultaneous and sequential cases analyzed in this article. That avoids the disconcerting “discontinuity” which otherwise occurs as the time interval \( t \) between the ejection of the two balls is reduced with a sudden change in the final speed of the rocket just as \( t \to 0 \).

Acknowledgment
Seth Rittenhouse suggested this problem in the form of a physics professor stranded on a frozen pond with two identical textbooks in his hands that he could throw with the goal of maximizing his recoil speed toward dry ground. It was presented at the Chesapeake Section of the American Association of Physics Teachers (http://csaapt.org/section-meetings.html) including a clicker question and was voted the best four-year-college-faculty talk at the Spring 2019 meeting.

References
