The Pursuit of a Plane by a Homing Missile

An elegant, purely algebraic solution has been presented by Kagan to the problem of finding the time that a missile takes to hit a plane flying on a level trajectory with constant velocity of magnitude \( u \). The missile is launched when the plane passes directly overhead at altitude \( R \). It always flies directly toward the plane at constant speed \( u \). Denote that time as \( t(90°) \), where \( 90° \) is the angle \( \theta_0 \) between the initial velocity vectors of the plane and missile. Equation (4) of Kagan’s article says that the time is \( t(90°) = Ru/(u^2 – v^2) \). The denominator is positive assuming \( u > v \), which is the condition for the missile to eventually catch up to the plane.

One can check that \( t(90°) \) is the average of the two values \( t(0°) = R/(u-v) \), corresponding to the case of the plane moving directly away from the missile, and \( t(180°) = R/(u+v) \) if the plane is headed directly toward the missile silo (located say on a mountain peak on the plane's trajectory). These two expressions state that the impact time is the initial distance divided by the speed of the missile relative to the plane, valid
because for these two special angles of 0° and 180°, the missile's trajectory is a straight line rather than the power-law curves plotted in Fig. 2 of Ref. 2.

Is it generally true \( t(\theta_0) \) is the average of \( t(\theta_0 + \Delta \theta_0) \) and \( t(\theta_0 - \Delta \theta_0) \), as was found to hold above for the case of \( \theta_0 = \Delta \theta_0 = 90° \)? No! For example, \( t(45°) \) is not equal to \( [t(0°) + t(90°)] / 2 \). Kagan's analysis can be generalized when the initial angle \( \theta_0 \) is not necessarily 90° by setting his \( \Delta X \) equal to \( -R \cos \theta_0 \) instead of to zero. The impact time then becomes

\[
t(\theta_0) = R \frac{u + v \cos \theta_0}{u^2 - v^2}.
\]

The reader is invited to check that this formula correctly reproduces the three values cited above for initial angles of 0°, 90°, and 180°. A remarkably short derivation of Eq. (1) is given in Ref. 3.

Based on Eq. (1), one can say that \( t(\theta_0) \) is the average of \( t(\theta_1) \) and \( t(\theta_2) \), where \( \theta_1 \) and \( \theta_2 \) are any two angles such that \( \cos(\theta_0) \) is the average of \( \cos(\theta_1) \) and \( \cos(\theta_2) \). For example, \( t(60°) = [t(0°) + t(90°)] / 2 \).

I thank Jim Vickers for drawing my attention to this issue.

References

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Kagan's response
Carl Mungan makes an interesting point about the impact time. I agree with his letter and would just like to add a couple of comments.

1) On the two sentences preceding Eq. (1): Finding the impact time for an arbitrary angle \( \theta_0 \) was suggested as a "homework" question in the online appendix of Ref. 1 (available at ftp://ftp.aip.org/epaps/phys_teach/EPHTEAH-51-009304, Appendix A, p. 7, second bullet item in question #2). Mungan’s solution and answer in Eq. (1) are exactly the expected solution and answer.

2) The curves that Mungan refers to in the last sentence of his second paragraph are also available in the online appendix (mentioned above) of Ref. 1. Specifically, the missile’s trajectory in the plane’s frame is given in Eq. (C9) and depicted in Fig. 11. The same trajectory in the lab (ground)-frame is given in Eq. (C10).

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