Radial descent of an energetically unbound spacecraft toward a comet

Regarding modeling\(^1\) the descent of the *Philae* lander from the *Rosetta* satellite onto comet 67P, an analytic solution can be found for the time \(t\) taken for a lander of point mass \(m\) to impact a comet of mass \(M \gg m\) and effective spherical radius \(R\), after the lander is launched radially toward the comet with initial speed \(v_0\) from radial distance \(r_0\). Conservation of energy gives the lander’s radially inward speed \(v\) when it has descended to radial distance \(r\) from the comet’s center as

\[
\frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \frac{1}{2}mv^2 - \frac{GMm}{r}
\]

where \(G\) is the universal gravitational constant. The values\(^1\) \(GM = 670 \text{ m}^3/\text{s}^2\), \(v_0 = 0.71 \text{ m/s}\), \(R = 2.4 \text{ km}\), and \(r_0 = 22.5 \text{ km}\) imply that the specific energy \(v_0^2/2 - GM/r_0\) of the lander is positive and hence one must modify the solution presented in the Appendix of Ref. 2 which applies to negative specific energies. Solve Eq. (1) for \(v\) and equate it to \(-dr/dt\) because \(r\) decreases as the lander approaches the comet,

\[
v = \sqrt{v_0^2 + 2GM \left( \frac{1}{r} - \frac{1}{r_0} \right)} = -\frac{dr}{dt}.
\]

Separate variables and integrate to get
The dimensionless parameter $\alpha$ can alternatively be written as $(v_0/v_{\text{esc}})^2 - 1$ where $v_{\text{esc}}$ is the escape speed at the lander’s starting position $r_0$, and thus $\alpha$ represents the fractional kinetic energy in excess of what is needed to escape from the comet. Since $\alpha$ here has the positive value 7.46 (so that the lander is gravitationally unbound), make the hyperbolic change of variables to $x$ specified by

$$\alpha r / r_0 = \sinh^2 x.$$  

Equation (3) then simplifies to

$$t = T \int_{x_1}^{x_2} 2 \sinh^2 x \, dx$$

where

$$\sinh x_2 = \sqrt{\alpha}$$

and $\sinh x_1 = \sqrt{\alpha R / r_0}$.

Here $T$ is a characteristic time given by

$$T \equiv \sqrt{\frac{R_0^3}{2GM \alpha^3}} = \frac{T_c}{\sqrt{8\pi^2 \alpha^3}},$$

where $T_c$ is the period of a circular orbit of the Rosetta satellite at radial distance $r_0$ above the comet. Using hyperbolic identities, the indefinite integral in Eq. (5) can be evaluated as

$$\int (\cosh 2x - 1) \, dx = \frac{1}{2} \sinh 2x - x = \sqrt{1 + \sinh^2 x} \sinh x - x$$

so that the final solution for the time required for the lander to impact the comet is

$$t = T \left[ f(R/r_0) - 1 \right]$$

where

$$f(z) = \sqrt{\alpha z (1 + \alpha z)} - \sinh^{-1} \sqrt{\alpha z}.$$  

(The inverse hyperbolic sine can alternatively be written in terms of a natural logarithm.3) Equation (8) can also be used to find the elapsed time $t$ at any point along the descent by replacing $R$ by $r$, and Eq. (2) can be used to calculate the speed $v$ at that point. These analytical calculations agree with the numerical results presented in Fig. 2 of Ref. 1. In particular, they predict that first impact onto the comet occurs at a speed of 1.00 m/s after a descent time of 7.32 h.

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1. P. R. Blanco, “Rosetta mission’s ‘7 hours of terror’ and Philae’s descent,” Phys. Teach. 53, 339–340 (Sept. 2015). Reference 9 in that paper should refer to Fig. 2 not Fig. 1.


3. Namely $\sinh^{-1} u = \ln \left( u + \sqrt{u^2 + 1} \right)$ so that Eq. (8) agrees with the results in http://www.mathpages.com/rr/s4-03/4-03.htm.