Entropic Damping of the Motion of a Piston

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The concept of an “entropic force” can be introduced by considering a familiar setup, namely a horizontal cylinder enclosing an ideal monatomic gas by a piston of cross-sectional area \( A \) and mass \( m \) that can slide without friction. The surrounding atmospheric pressure \( P_{\text{atm}} \) keeps the piston from flying out of the cylinder. The cylinder and piston have negligible heat capacity (compared to the gas), but the gas is not thermally insulated from the surroundings at room temperature \( T_R \). Ignore any viscosity or turbulence of the enclosed gas or surrounding air. Two specific and illustrative situations are analyzed here. In the first, the piston is massless, \( m = 0 \). The piston is temporarily held in place by a pin while the gas is quickly adjusted to initial pressure \( P_i = P_{\text{atm}} \) and temperature slightly larger than that of the room, say \( T_i = 1.1 T_R \), using a heater and regulator. The piston is then released from rest, \( v_i = 0 \). In the second case, the piston has inertia, \( m > 0 \), and the gas is initially in both mechanical and thermal equilibrium with the surroundings so that \( P_i = P_{\text{atm}} \) and \( T_i = T_R \). The piston is now given a quick inward push, \( v_i < 0 \). In both situations, the aim is the same: Describe the subsequent evolution of the system.

![Diagram of the system](image)

**Overview of key ideas**

The general setup for both cases is sketched in Fig. 1. The two situations are similar in that the system will evolve toward equilibrium with the surroundings so that \( P_i = P_{\text{atm}} \) and \( T_i = T_R \). However, there are important differences between the two problems. In the second situation, the inertia of the piston causes it to act like a mass on a spring. Accordingly, the piston will begin to oscillate. What is less obvious is that the oscillations will be damped, despite the fact that there are no frictional or viscous forces present. Perhaps even more surprising is that in the first situation the motion of the piston will be overdamped and thus it will not oscillate at all; instead the gas will exponentially contract in volume as heat \( Q \) is transferred out of the gas. Also not obvious is that an analytic solution can be found for this over-damped case, which is why it is analyzed first. In the second situation, in contrast, it is necessary to resort to a numerical solution, but that can be neatly accomplished in a spreadsheet.

The motion of the piston in both cases is damped by an entropic force. The general idea behind this dissipative mechanism is as follows. Heat transfer (across the surfaces of the cylinder and piston) between the gas and the atmosphere is irreversible whenever their temperatures differ by more than an infinitesimal amount. The entropy of the universe must then monotonically increase, regardless of whether the piston is moving inward or outward, and regardless of whether the gas is hotter than the surroundings or vice versa. However, entropy cannot keep increasing forever without limit! It must have a maximum, which coincides with a final equilibrium state of the system. Thus the piston must eventually come to rest.

However, calling this damping an “entropic force” can be confusing for students because entropy is not a direct force agent. For that matter, heat transfer does not automatically result in a dissipative force. (One could imagine continuously adjusting the temperature of the surroundings so that it differs only infinitesimally from that of the gas. Then the heat transfer would be reversible and there would be no damping.) The actual force on the piston is due to the gauge pressure of the gas. However, changes in the temperature and volume of the gas (of fixed number of moles \( n \)) are connected via the ideal gas law to changes in its pressure. Although it is not evident from thinking about generic “energy” sloshing between the gas, piston, and atmosphere, the second law of thermodynamics demands that the gas pressure must vary in a manner that leads to damped motion of the piston. In this sense, the dissipation is entropy driven.

To mathematically demonstrate what happens, all that is needed are the two relevant equations of motion (the first law of thermodynamics for the gas and Newton’s second law for the piston). Although a simultaneous analytic solution of these two equations does not in general exist, even introductory students can integrate them numerically using the Euler-Cromer method in Excel. This article thereby provides an accessible illustration of entropic forces, an important topic that is rarely covered in undergraduate physics courses with the possible exception of the shrinking of stretched rubber bands upon heating.

**General equations of motion**

In Fig. 1, the distance from the fixed left end of the cylinder to the piston is defined to be \( x \), with an initial value of \( x_i \). The
main goal is to find the subsequent position of the piston as a function of time, \( x(t) \). One also learns how the pressure \( P(t) \) and temperature \( T(t) \) of the confined gas vary as functions of time. The piston is assumed to move slowly enough (compared to the speed of sound) that all portions of the gas have the same pressure and temperature throughout its volume. (The room is taken to be large enough that it can be treated as a bath at constant pressure \( P_{\text{atm}} = 101 \text{ kPa} \) and temperature \( T_R = 300 \text{ K} \).) Further assume that the temperature of the gas is always close enough to room temperature that Newton’s law of cooling applies. That is, the thermal current from the gas to the surroundings is linearly proportional to their temperature difference, \( dQ/dt = k n R (T - T_R) \). Since the heat transfer is expected to scale with the amount of gas, \( n \) has been factored out of the right-hand side, along with the universal gas constant \( R = 8.314 \text{ J K}^{-1} \text{ mol}^{-1} \) that always multiplies \( n \) in the following equations, and thus the heat transfer coefficient \( k \) has dimensions of reciprocal time. The details of the heat mechanism need not concern us but are dominated by thermal conduction in an experiment such as the damping of a steel ball oscillating in the glass neck of a Rüchardt flask.\(^2\)\(^3\) Meanwhile, the work that the gas does on the piston in pushing it rightward a differential distance \( dx \) is \( dW = P A dx \), which becomes \( dW = n R T dx / x \) using the ideal gas law. As a result of these heat and work outputs (indicated by adding minus signs to \( dW \) and \( dQ \)), the internal energy of the monatomic gas changes by \( dU = 1.5 n R dT \). The first law of thermodynamics thus implies

\[
1.5 n R dT = -\frac{n R T}{x} \, dx - k n R (T - T_R) \, dt.
\]

Divide through by the time increment \( dt \) during which these changes occur to get

\[
\dot{T} = \frac{2}{3} k T_R - \frac{2}{3} \left( \frac{\dot{v}}{x} + k \right) T,
\]

where \( \dot{v} \) is the rightward velocity of the piston and the overdot on the left-hand side indicates a time derivative. On the other hand, Newton’s second law for the piston is

\[
\frac{n R T}{x} - P_{\text{atm}} A = ma,
\]

where \( a \) is the rightward acceleration of the piston. This equation can be rewritten as

\[
\dot{v} = \frac{n R T}{m x} - \frac{P_{\text{atm}} A}{m}.
\]

**Solution to problem 1**

In the first problem, when the piston is massless, Eq. (3) immediately implies that

\[
T = \frac{P_{\text{atm}} A}{n R} x.
\]

The net force on a massless object must be zero. Accordingly, Eq. (5) says that the gas pressure always balances atmospheric pressure. If that were the full story, it would be hard to see why the piston should begin to move. In fact, the two pressures will differ infinitesimally as heat is transferred out of the gas and one must consider the first law of thermodynamics to make further progress toward the solution. Substituting Eq. (5) into both the left and right sides of Eq. (2) to eliminate \( T \), one finds

\[
\frac{dx}{x - x_f} = -0.4 k \, dt,
\]

where the final position of the piston is \( x_f = n R T_R / P_{\text{atm}} A \) according to the ideal gas law. Integrating Eq. (6) and imposing the initial conditions, the solution has the overdamped form

\[
x(t) = (x_i - x_f) e^{-0.4 k t} + x_f,
\]

with a damping constant of 0.4\( k \). Why is this situation overdamped? Recall that a mass \( m \) on a spring \( k \) subject to a linear damping force \(-\dot{b} v \) is overdamped if \( b / 2 m > \sqrt{k / m} \). But that inequality is guaranteed to be satisfied for small\(^6\) enough \( m \) for any given values of \( b \) and \( k \).

For the purposes of comparing this solution with that of the second problem, as many system parameters as possible will be chosen to be common between the two situations. Take the piston to have area \( A = 0.01 \text{ m}^2 \) and let its final equilibrium position be \( x_f = 0.1 \text{ m} \). The number of moles of gas in the cylinder is then \( n = P_{\text{atm}} x_f A / R T_R = 0.04 \text{ mol} \). For reasons explained in the Appendix, the value of the heat transfer coefficient is taken to be \( k = 320 \text{ s}^{-1} \). Thus \( n R k = 100 \text{ W/K} \), which is comparable to the thermal conductance of the glassware typically used in Rüchardt experiments.\(^2\)

The ideal gas law implies that \( x_i = x_f T_i / T_f = 0.11 \text{ m} \). Figure 2 is a plot of Eq. (7) for these numerical values. The piston starts 11 cm away from the left-hand end of the cylinder and exponentially approaches a position 10 cm away from it. It is \( e^{-1} = 0.37 \text{ cm} \) away from its final position after one decay time of \( 1/0.4 k = 7.8 \text{ ms} \). The acceleration of the piston is nonzero for this graph, despite the fact that the net force on the piston is zero; in Newton’s second law, \( a = F_{\text{net}} / m \) can be nonzero for a massless object! However, prior to \( t = 0 \), the piston had zero velocity and acceleration; the discontinuity in both \( v \) and \( a \) at \( t = 0 \) is due to the release of the pin holding the piston at that instant in time.
Solution to problem 2

A flowchart to numerically solve Eqs. (2) and (4) simultaneously in a spreadsheet is presented in Fig. 3. The gas pressure at any point in the solution can be calculated as $P = nRT/Ax$, where $n$ was calculated above to be approximately 0.04 mol. In this second problem, the gas is initially at room temperature, $T_i = 300$ K, and at atmospheric pressure, $P_i = 101$ kPa. Since these are also the final temperature and pressure of the gas, the piston must start and end at the same position, $x_i = x_f = 0.1$ m. For a metal piston with an area of 100 cm$^2$, a thickness of say 1 mm, and a density of 10 g/cm$^3$, its mass is $m = 0.1$ kg. To keep the temperature from oscillating by more than a few percent and ensure Newton’s law of cooling always holds, give the piston a moderate leftward tap, say $v_i = -1$ m/s. In the Appendix, $k$ is found to be 320 s$^{-1}$.

Figure 4 uses Excel to plot $x(t)$ for these values. It is important to choose the step size $dt$ in the flowchart in Fig. 3 to be small enough to achieve convergence; as a check, the value of $dt$ is halved until no further change in the graph is seen. Here, a step size of $dt = 0.1$ ms is found to suffice. One sees that $x$ starts at 10 cm with a slope of $-1$ m/s, experiences underdamped oscillations, and returns to rest at 10 cm in a fraction of a second, just like a mass on a spring that is given an impulsive kick. By finding the times at which the curve crosses the horizontal axis and fitting a straight line to them versus the number of half-cycles elapsed, the period of the oscillations is measured to be 15.9 ms. This value agrees with the Rüchardt prediction of

$$\tau = 2\pi x_i \sqrt{\frac{m}{k nRT}} = 15.3 \text{ ms},$$

where $\gamma = 5/3$ is the ratio of the isobaric and isochoric heat capacities for a monatomic ideal gas, thereby explaining the success of Rüchardt’s method for measuring $\gamma$ despite its neglect of damping.

Summary

The motion of a frictionless piston with a confined ideal gas on one side and atmosphere on the other side has been analyzed, assuming the gas and the surroundings are in both mechanical and thermal contact with each other. Despite the absence of any explicit dissipative mechanism, the motion of the piston is damped due to the irreversible heat transfer between the gas and the atmosphere. (The entropy change of the universe can be explicitly calculated to be positive, confirming this conclusion.) If the mass of the piston is large (compared to the mass of the confined gas and surrounding air that are entrained by it), then its oscillations are underdamped with a period that agrees with Rüchardt’s method for measuring the ratio of the heat capacities of a gas. On the other hand, if its mass is small, then the system is overdamped: when displaced and released from rest, the piston exponentially returns to its equilibrium position.

Appendix: Choosing a value for $k$

The heat transfer coefficient $k$ needs to be large enough in value that the entropic damping will be significant, but how big is that? Absent any knowledge of the exact heating mechanisms, a way to estimate it is to recast Eqs. (2) and (4) in dimensionless form to see how they scale. Define normalized variables relative to the initial values as $\tilde{x} = x/x_i, \tilde{T} = T/T_i$, and $\tilde{P} = P/P_i$. These are all necessarily positive because they set the gas volume, absolute temperature, and absolute pressure. Time cannot be similarly normalized because it starts out zero; instead a logical choice is to define $\tilde{t} = kt$. Then $\tilde{v} = d\tilde{x}/d\tilde{t} = v/x_i k$. Equation (2) written
in terms of these new variables has no remaining parameters in it,

\[ \frac{d\tilde{T}}{dt} = \frac{2}{3} \tilde{T}_R - \frac{2}{3} \left( \frac{\tilde{v}}{\tilde{x}} + 1 \right) \tilde{T}. \]  

(9)

However, Eq. (4) retains one scaling parameter \( \varepsilon \) in it,

\[ \frac{d\tilde{v}}{dt} = \varepsilon \left( \frac{\tilde{T}}{\tilde{x}} - 1 \right). \]  

(10)

where \( \varepsilon = \frac{P_{\text{atm}} A/m x_i}{x_i} \) is the ratio of the atmospheric pressure force to the thermally driven damping force on the piston. Setting \( \varepsilon = 1 \) ensures the entropic dissipation will be significant, which implies \( k = (P_{\text{atm}} A/m x_i)^{\frac{1}{2}} \approx 320 \text{ s}^{-1} \) for the choice of parameters specified in the solution to the second problem.

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References
1. The present setup thus contrasts with the analysis of thermally insulated piston problems, such as in C. E. Mungan, “Irreversible adiabatic compression of an ideal gas,” *Phys. Teach.* 41, 450–453 (Nov. 2003).
6. Technically \( m \) is nonzero even when the piston is massless, because one should include a fraction (on the order of a third to a half) of the mass of the gas (as well as some of the surrounding air) in the oscillations. See H. L. Armstrong, “The oscillating spring and weight: An experiment often misinterpreted,” *Am. J. Phys.* 37, 447–449 (April 1969).
7. Further discussion and analysis of the underdamping are in the Online Supplement under the “References” tab at *TPT Online* [supplementary material] http://dx.doi.org/XXX.

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The piston in the second problem exhibits damped oscillations. Further details of this motion are presented here. In accordance with the solution of the second problem, set $\varepsilon = 1$ and $\tilde{T}_R = 1$ and drop the tildes on the variables in Eqs. (9) and (10) to reduce clutter. Equation (10) now rearranges into

$$ T = x\dddot{x} + x $$

whose time derivative is

$$ \ddot{T} = x\dddot{x} + x\dddot{x} + \dot{x}. $$

Substitute Eqs. (11) and (12) into (9) and simplify to get

$$ 1.5x\dddot{x} + 2.5x\dddot{x} + x\dddot{x} + 2.5\dot{x} + x = 1. $$

To eliminate the inhomogenous term, substitute $x = 1 + z$ where $z(t)$ is the small displacement of the piston from equilibrium. Then neglect the nonlinear terms (i.e., products of $z$ and its derivatives) in Eq. (13) to end up with the final result

$$ 1.5\dddot{z} + 2.5\dddot{z} + 2.5\dot{z} + z \approx 0. $$

The initial conditions for problem 2 are $z(0) = 0$, $\dot{z}(0) = (-1 \text{ m/s}) / (0.1 \text{ m})(320 \text{ s}^{-1}) = -1/32$, and $\ddot{z}(0) = 0$ according to Eq. (10). Equation (14) can thus be numerically integrated by typing into Mathematica the expression

$$ s = \text{NDSolve}[\{1.5 z''''[t] + z''[t] + 2.5 z'[t] + z[t] == 0, z[0] == 0, z'[0] == -1/32, z''[0] == 0\}, z, \{t, 0, 32\}] $$

and then plotting it, noting that $t$ converts to milliseconds by dividing it by $k = 0.32 \text{ ms}^{-1}$ and that $z$ converts to centimeters by adding 1 to it and then multiplying the result by $x_i = 10 \text{ cm}$, using the command

$$ \text{Plot[Evaluate[10*(1 + z[t*0.32]) /. s]}, \{t, 0, 100\}, \text{PlotRange -> \{\{0, 100\}, \{9.7, 10.3\}\}}] $$

to get the blue curve in Fig. 5. This blue curve is in excellent agreement with the red curve repeated from Fig. 4 which is an exact numerical solution of the differential equations (in contrast to the neglect of the nonlinear terms in this Supplement). To verify that the result is approximately of the classic underdamped form, the overlaid green curve in Fig. 5 is a plot of the function

$$ z(t) = -Xe^{-\beta t} \sin(2\pi t / \tau) $$

with $\tau = 15.9 \text{ ms}$ taken from the discussion preceding Eq. (8), $X = -v_i \tau / 2\pi = 0.25 \text{ cm}$ deduced from the initial conditions, and $\beta = k / 10$ chosen to get a reasonable fit in Fig. 5 by eye. Although this value of $\beta$ does not accord with the initial conditions (only $\beta = 0$ would!) it predicts a small value for the initial acceleration in the units of Fig. 5, namely $\dddot{x}(0) = -2\beta v_i = 0.0064 \text{ cm} / \text{ms}^2$. 
The upshot is that Eq. (14) looks similar to the classic damped oscillator form $m\ddot{x} + bx + \kappa x = 0$ except that it includes an explicit jerk term $da/dt = \dddot{x}$. (Note that if one solves the classic damped oscillator equation for the acceleration $a$ and differentiates it, one also finds a nonzero jerk.) It is thus appropriate to describe these oscillations as being “underdamped.” Similar terminology is used in Refs. 10 and 11 for motion that is also not exactly of the classic damped oscillator form.

**Figure 5.** Comparison of three different solutions for the motion of the piston in problem 2: the red curve is the exact result from Fig. 4, whereas the blue and green curves are plots of a numerical integration of the approximate Eq. (14) and of Eq. (15), respectively, after converting $z$ back to $x$.

**References**