Optimizing the Launch of a Projectile to Hit a Target

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Some teenagers are exploring the outer perimeter of a castle. They notice a spy hole in its wall, across the moat a horizontal distance x and vertically up the wall a distance y. They decide to throw pebbles at the hole. One girl wants to use physics to throw with the minimum speed necessary to hit the hole. What is the required launch speed v and launch angle θ above the horizontal?

The situation is sketched in Fig. 1, where x and y are measured from the launch point near the girl's right shoulder. Standard kinematics for projectile motion imply that the flight time is

$$t = \frac{x}{v\cos\theta} \tag{1}$$

and

$$y = (v\sin\theta)t - \tfrac{1}{2}gt^2, \tag{2}$$

where $g = 9.8\ \mathrm{m/s^2}$ is Earth's surface free-fall acceleration. (Air resistance is assumed negligible.) Substituting Eq. (1) into (2) and rearranging leads to

$$v = \sec\theta\sqrt{\frac{gx/2}{\tan\theta - \tan\phi}}, \tag{3}$$

where $\tan\phi = y/x$ from Fig. 1. Take the derivative of Eq. (3) with respect to the launch angle and set it equal to zero to minimize the launch speed. As shown in the online appendix,[1] the result simplifies to the quadratic equation

$$\tan^2\theta - 2\tan\phi\tan\theta - 1 = 0, \tag{4}$$

whose positive root is

$$\tan\theta = \tan\phi + \sec\phi. \tag{5}$$

As further shown in the appendix, this equation has the simple solution[2]

$$\theta = 45^\circ + \phi/2. \tag{6}$$

Operationally, the girl could determine this optimal launch angle as follows. Holding the stone in her right hand, point her left arm directly at the hole in the wall. Next bring her right arm upward until it points perpendicular to her left arm. Then slightly rotate her left arm upward until it bisects the angle between the ground and her right arm. She can then release her right arm from its position, while her left arm is held fixed as the bore line along which the stone in her right hand must be thrown.

Substituting Eq. (5) into (3), the appendix shows that the minimum launch speed is

$$v = \sqrt{g(y+r)}, \tag{7}$$

where $r = (x^2 + y^2)^{1/2}$ is the distance from the launch point of the stone to the hole in the wall, as shown in Fig. 1. Equations (6) and (7) have the expected limiting forms for $y = 0$ (namely $\theta = 45^\circ$ and $v = \sqrt{gx}$, which are the familiar formulas for maximum range) and for $x = 0$ (namely $\theta = 90^\circ$ and $v = \sqrt{2gy}$, corresponding to a vertical throw). As a realistic example, if $x = y = 10$ m, then $\theta = 67.5^\circ$ and $v \approx 15.4$ m/s $= 34.3$ mph.

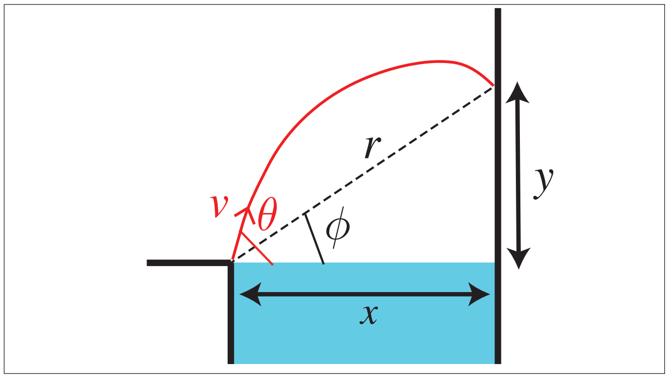

Fig. 1. Trajectory of a stone launched with optimal speed v and angle θ such that it passes through a hole located at rectangular coordinates (x,y) relative to the launch point. In polar coordinates the hole is located at (r,ϕ).

Intuitively, minimizing v with respect to θ for fixed x and y should be equivalent to maximizing x with respect to θ for fixed y and v. To prove it, solve Eq. (7) for r, square the result and equate it to $x^2 + y^2$, and then isolate x to get

$$x = \frac{v^2}{g}\sqrt{1 - \frac{2gy}{v^2}}. \tag{8}$$

This formula indeed gives the maximum range of a projectile launched from height $h = -y$ relative to the final level.[3] It can alternatively be written in the compact form $x = v_{\mathrm{ave}}^2/g$, where v_{ave} is the geometric average of the initial speed v and the final speed $(v^2 - 2gy)^{1/2}$ from energy conservation. Equation (8) could be used to find the maximum horizontal distance a basketball player could be away from the hoop and hope to score if the ball is thrown with a given speed. As shown in the appendix, the final vertical velocity component is

$$v_{\mathrm{fy}} = (y - r)\sqrt{\frac{g}{2r}}, \tag{9}$$

which is negative, and thus the ball will *fall* into the hoop. For all positive values of x and y, the projectile passes the vertex of its parabolic path *before* it reaches the target, as sketched in Fig. 1.

To experimentally verify these concepts, a PASCO Mini Launcher[4] was used to project a standard 16-mm diameter steel solid ball. The long-range "third click" trigger setting was used at a launch angle of $\theta = 35.0^\circ \pm 0.5^\circ$ above the horizontal. The launch speed v is simply related to the range R measured at a final height equal to the launch height marked on

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the muzzle of the launcher. The range for about 10 trials was found to be $R = 2.36 \pm 0.02$ m, where the uncertainty includes both the repeatability of the result and the measurement error in the distance. Consistent with that uncertainty in range, the target used was a 4-cm diameter plastic cup held in the clamp of a ring stand. The clamp was raised until the rim of the cup was at a vertical height of $h = 30.8 \pm 0.2$ cm above the launch mark. As shown in the appendix, the cup then needs to be positioned a horizontal distance D away from the launch mark on the muzzle given by

$$D = 0.5R\left(1+\sqrt{1-4hR^{-1}\cot\theta}\right), \tag{10}$$

which works out to be 1.78 ± 0.04 m. The ring stand was moved along the floor to center the cup at that distance from the launcher, and the ball was found to consistently fall into it. However, it would bounce out of the 3-cm deep cup. To prevent that from happening, the bottom of the cup was cut off, and the ball would then pass through it. Reproducing this kind of setup could be a fun challenge for introductory students to perform in lab.

References

1. See the "Supplemental" tab at *TPT Online,* http://dx.doi.org/10.1119/1.5011825, for the appendix.
2. Compare to Eq. (1) in H. Van Dael and H. Bert, "Range of a projectile," *Am. J. Phys.* **47**, 466–467 (May 1979) and to Eq. (21) in R. A. Brown, "Maximizing the range of a projectile," *Phys. Teach.* **30**, 344–347 (Sept. 1992).
3. See Eq. (4) in W. S. Porter, "The range of a projectile," *Phys. Teach.* **15**, 358 (Sept. 1977); Eq. (9) in D. B. Lichtenberg and J. G. Wills, "Maximizing the range of the shot put," *Am. J. Phys.* **46**, 546–549 (May 1978); Eq. (5) in S. K. Bose, "Maximizing the range of the shot put without calculus," *Am. J. Phys.* **51**, 458–459 (May 1983); and Eq. (6) in B. Bušić, "A simple solution for maximum range of projectile motion," *Phys. Teach*. **51**, 52 (Jan. 2013).
4. https://www.pasco.com/prodCatalog/ME/ME-6825_mini-launcher/.

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Online Appendix for "Optimizing the Launch of a Projectile to Hit a Target"

Derivations of six equations in the main text are presented here.

Derivation of Eqs. (4) and (5)

Divide the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ through by $\cos^2\theta$ to get $1+\tan^2\theta = \sec^2\theta$ so that Eq. (3) can be rewritten as

$$\upsilon^2 \propto \frac{1+\tan^2\theta}{\tan\theta - \tan\phi} \tag{A1}$$

and thus

$$\frac{d(\upsilon^2)}{d\theta} = 0 \quad \Rightarrow \quad \frac{2\tan\theta\sec^2\theta}{\tan\theta - \tan\phi} = \frac{(1+\tan^2\theta)\sec^2\theta}{(\tan\theta - \tan\phi)^2}\,. \tag{A2}$$

Cross-multiplying leads to

$$2\tan\theta(\tan\theta - \tan\phi) = 1+\tan^2\theta \tag{A3}$$

which rearranges into Eq. (4). That equation can be solved using the quadratic formula to get

$$\tan\theta = \frac{2\tan\phi \pm \sqrt{4\tan^2\phi + 4}}{2}\,. \tag{A4}$$

Choosing the upper sign to get a positive solution and again using the identity $1+\tan^2\phi = \sec^2\phi$ leads to Eq. (5).

Derivation of Eq. (6)

Equation (5) can be solved by separately manipulating the two sides of that equation. The left-hand side is

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{1-\cos 2\theta}{\sin 2\theta} \tag{A5}$$

invoking the double-angle formulas for cosine and sine in the last step. Next, using complementary angles, Eq. (A5) can be rewritten as

$$\tan\theta = \frac{1-\sin(90^\circ - 2\theta)}{\cos(90^\circ - 2\theta)} = \frac{1+\sin(2\theta - 90^\circ)}{\cos(2\theta - 90^\circ)}\,. \tag{A6}$$

On the other hand, the right-hand side of Eq. (5) is

$$\sec\phi + \tan\phi = \frac{1+\sin\phi}{\cos\phi}\,. \tag{A7}$$

Equate the right-hand sides of Eqs. (A6) and (A7) to deduce

$$2\theta - 90^\circ = \phi \tag{A8}$$

which rearranges into Eq. (6).

Derivation of Eq. (7)

Again using the identity $1+\tan^2\theta = \sec^2\theta$, Eq. (3) can be rewritten as

$$\upsilon = \sqrt{\frac{(g/2)(x^2 + x^2\tan^2\theta)}{x\tan\theta - x\tan\phi}}. \tag{A9}$$

But from Fig. 1, $\tan\phi = y/x$ and $\sec\phi = r/x$ so that Eq. (5) becomes

$$\tan\theta = \frac{y+r}{x}. \tag{A10}$$

Substituting these expressions for $\tan\phi$ and $\tan\theta$ into Eq. (A9) gives

$$\upsilon = \sqrt{\frac{(g/2)(x^2 + y^2 + 2yr + r^2)}{r}} \tag{A11}$$

which, after replacing $x^2 + y^2$ with r^2 in the numerator, simplifies to Eq. (7).

Derivation of Eq. (9)

The *y*-component of the final velocity of the projectile is $\upsilon_{\rm fy} = \upsilon\sin\theta - gt$ into which Eq. (1) is substituted to get

$$\upsilon_{\rm fy} = \frac{1}{\upsilon\sec\theta}\left(\upsilon^2\tan\theta - gx\sec^2\theta\right) = \frac{1}{\upsilon\sqrt{1+\tan^2\theta}}\left(\upsilon^2\tan\theta - gx - gx\tan^2\theta\right) \tag{A12}$$

using the now familiar trigonometric identity $1+\tan^2\theta = \sec^2\theta$ in the second step. Next, substitute Eqs. (7) and (A10) to obtain

$$\upsilon_{\rm fy} = \frac{-x^2\sqrt{g}}{\sqrt{y+r}\sqrt{x^2 + (y+r)^2}} \tag{A13}$$

Finally, substitute $x^2 = r^2 - y^2$ to get Eq. (9).

Derivation of Eq. (10)

When the ball lands at the same height at which it is launched with speed υ and angle θ, then Eq. (1) can be substituted into Eq. (2) with range $x = R$ and final height $y = 0$ to obtain

$$\upsilon = \sqrt{gR\csc 2\theta} \tag{A14}$$

using the double-angle formula for sine. Alternatively, this launch speed can be measured using photogates positioned at the end of the muzzle but that requires additional equipment: careful alignment of two computerized photogates gave a speed υ within 2% of the value predicted by Eq. (A14). Next, again substitute Eq. (1) into (2) but this time with $x = D$ and $y = h$, and then solve it for *D* using the quadratic formula with the positive choice of sign to find

$$D = \frac{\upsilon^2 \sin 2\theta}{2g}\left(1 + \sqrt{1 - \frac{2gh}{\upsilon^2 \sin^2\theta}}\right) \tag{A15}$$

using the sine double-angle formula. Substituting Eq. (A14) into (A15) results in Eq. (10). The muzzle speed varies by 8% as θ varies from 0° to 90° for the Pasco launcher according to its instruction manual, and so it is important that both R and D are measured at the same value of θ.