Consider a chain of length \( L \) that hangs in a U shape with end A fixed to a rigid support and free end E released from rest starting from the same initial height (call it \( y = 0 \)) as A. Figure 1 sketches the chain after end E has fallen a distance \( y \). Points O and A are assumed to be close enough to each other and the chain flexible enough that the radius of curvature \( r \) at the bottom point C can be taken to be negligibly small (compared to the length of the chain). The problem is to compare the speed of descent \( v(y) = dy/dt \) of the free end E of the chain to the speed \( v_{\text{free}}(y) = \sqrt{2gy} \) of a free-falling point mass that has descended the same distance \( y \). If \( v(y) > v_{\text{free}}(y) \) for all \( y > 0 \), then, in a race to fall any arbitrary distance \( Y \) (where \( 0 < Y < L \)), the chain end E will always beat a simultaneously released point mass, because the fall time \( t \) for E will be shorter than \( t_{\text{free}} \) for the point mass,

\[
 t = \int_0^y \frac{dy}{v(y)} < t_{\text{free}} = \int_0^y \frac{dy}{v_{\text{free}}(y)}.
\]

(1)

Experimentally, this outcome of the race is observed in a “Veritasium” video.\(^1\)

**Related previous work**

This problem has appeared in mechanics textbooks\(^2\) dating back to at least 1897. However, the earliest discussion of it in a pedagogical journal is in a pair of letters to the editor\(^3\) in 1951, with the first systematic analysis appearing a few years later.\(^4\) But it is only with the publication of Calkin and March’s first paper\(^2\) on the topic in 1989 that widespread interest led to a large number of subsequent articles continuing up to the present day.

There have been two main schools of thought about the solution to this problem, with both sides claiming implicitly or even explicitly that their approach is obviously correct. This situation should give caution to any instructor who is tempted to quickly explain the physics of this system to their students without justifying the assumptions and models underlying their explanation.\(^5\) One approach to the chain drop problem has been to assume that section DE in Fig. 1 is in free fall so that \( v(y) = v_{\text{free}}(y) \). For example, a recent paper that makes and then appears to theoretically validate this assumption is by de Sousa and Rodrigues.\(^6\) To reemphasize how difficult it can be to sort out this matter, I challenge the readers to find the flaw in their theoretical analysis before they look up my online appendix\(^7\) presenting it. It turns out that making this free fall assumption is equivalent to assuming that all of the downward translational kinetic energy is lost as each successive link moving with speed \( v \) at point D in Fig. 1 turns the bottom corner through point C of the fold in the chain and comes to rest at point B. Presumably most of the kinetic energy lost by this bottom-turning link would in this view be dissipated as thermal energy as it inelastically rubs or jerks against the stationary link nearest point B of the chain.\(^8\) The situation does not seem apparently different than what happens at the bottom of a chain falling into a heap on a table. In fact, several articles explicitly make such a comparison between the fall of a folded and an unfolded chain.\(^9,10\) The rest of the lost downward translational kinetic energy of individual links is converted into overall rotational swinging and vibrational oscillations of the chain seen in video analysis.\(^11–13\)

Despite the plausibility of these ideas about mechanical energy dissipation and about the free fall of section DE of the chain, careful comparison between simulations and experiments reveals that most of the kinetic energy of the falling chain is not lost but is instead “concentrated” into the increasingly shorter falling section, leading to \( v(y) > v_{\text{free}}(y) \), which is the second school of thought. In fact, to a good approximation none of the mechanical energy of the system is lost\(^2,4\) although I am not alone\(^14\) in taking issue with authors\(^11\) who consider this conservation of mechanical energy to be easy to justify theoretically. Another system that exhibits conservation of mechanical energy is the bungee jumper,\(^15,16\) although that is less surprising because the cord can be modeled as an elastic spring.

In view of these subtleties associated with energy, there is merit to developing alternative methods of analyzing the dynamics of the falling chain. An elegant Lagrangian approach has been articulated by Wong and Yasui,\(^17\) but that is too sophisticated for an introductory course. The present paper uses only Newton’s laws to derive the solution to the problem. No other article, to the best of my knowledge, fully derives the results using forces and accelerations alone at a level of treatment accessible to an introductory calculus-based mechanics
course. Only at the end of the present paper is mechanical energy discussed, where it is **deduced** rather than **assumed** to be conserved.

**Theoretical analysis**

Sections AB and DE of the chain in Fig. 1 are approximated in this article as starting out and subsequently remaining exactly vertical. For a theoretical infinitely flexible chain, that will only be true if the radius of curvature $r$ of section BD is zero (i.e., if points A and O in Fig. 1 lie on top of each other). For a real chain composed of an odd number of links, sections AB and DE will initially be vertical if points A and O are separated by a length of exactly one link, as illustrated in Fig. 11 of Ref. 11. In contrast, for larger horizontal distances between points A and O, sections AB and DE will initially adopt an approximately catenary shape, and photographs of the subsequent motion of the chain after releasing end E can be found in Refs. 12 and 13.

Whereas section DE of the chain falls as a unit with one overall vertically downward speed $v$ (and corresponding downward acceleration of magnitude $a$), the links of the chain between points B and D do not move in the directions tangential to the instantaneous shape of the chain in Fig. 1. For example, point C has a vertical velocity component of $v/2$ downward in addition to its horizontal velocity component of $v/2$ rightward. It is analogous to the velocities of points on the rim of a tire rolling downward without slipping along line AB: point B then represents the bottom of the tire at rest relative to the line, point D the top of the tire moving downward at speed $v$, and point C the front-most point of the tire moving downward and rightward with each of those velocity components equal to half of $v$.

It is helpful to analyze the motion of the chain in a frame of reference moving downward at speed $v/2$, as in Fig. 2. (This frame is accelerating vertically downward at the time derivative $a/2$ of its speed.) In this frame co-moving vertically with point C, section BD of the chain is rotating with speed $v/2$ around the stationary center F of a semicircle of radius $r$, like a rope on an ideal pulley. Thus its tangential acceleration is $a_c = a/2$. The mass of this semicircular section is $m = \lambda \pi r$, where $\lambda$ is the linear mass density of the chain. Denote the magnitudes of the vertically upward tension on ends B and D of this section as $T_B$ and $T_D$, respectively. The tangential component of Newton's second law for section BD of the chain becomes

$$
T_B - T_D = ma_c = \lambda \pi r a/2.
$$

(2)

This equation could alternatively be deduced by writing down Newton's second law in rotational form in terms of the torques on, moment of inertia of, and angular acceleration of section BD about point F in Fig. 2. In the limit that the chain makes a tight turn so that $r \to 0$, it follows from Eq. (2) that $T_B = T_D$. Approximately the same tension, call its magnitude $T_{BD}$ is exerted all the way around section BD of the chain (just as is true for a rope that does not slip on a pulley if the mass of both the pulley and of the section of the rope on the pulley are negligible and there is no friction at the axle of the pulley). Even though point B is stationary while point D is moving downward at speed $v$ in the lab frame, the upward tension exerted on point B by the chain above it is equal to the downward tension exerted on point D by the rotating section of chain below it, in agreement with a second “Veritasium” video that follows up on Ref. 1. This downward tension force on D in Fig. 1, over and above gravity, which acts both on the chain section DE and on a falling point mass, is what causes the chain end E to outrace the point mass.

To find a quantitative expression for the speed of the free end E, consider the vertical component of Newton's second law in the lab frame of reference for this semicircular loop of chain, as illustrated in Fig. 3. In addition to the upward tension $T_{BD}$ acting on each of its ends, the downward gravitational force on the section is $mg = \lambda \pi r g$. The acceleration of each infinitesimal segment of the loop is the sum of three terms: the tangential acceleration $a_c$ (as shown in Fig. 2), the overall vertically downward acceleration $a/2$ of the falling loop, and the centripetal acceleration $a_c = (v/2)^2/r$ as the segment rotates at speed $v/2$ around the center point F. To find the net acceleration of the entire section BD, average together the vector accelerations of each individual segment and then multiply that average acceleration by the total mass $m = \lambda \pi r$ of the section. (Equivalently, integrate the product of the vector acceleration and infinitesimal mass of each segment over the entire section.) By symmetry, the average vertical component of the tangential acceleration is zero, as is the average horizontal

![Fig. 2. Forces ($T_B$ and $T_D$) and acceleration ($a_c$) that have a net tangential component for chain section BD of mass $m$ in the vertically co-moving frame. That section makes a semicircular loop centered on point F.](image1)

![Fig. 3. Forces ($T_{BD}$ and $mg$) and accelerations ($a/2$ and $a_c$) that have a net vertical component for chain section BD of mass $m$ in the lab frame of reference. That section is rotating about point F with speed $v/2$. It is semicircular with radius $r$ such that any point on the section is located at angle $\theta$ varying from $-\pi/2$ to $\pi/2$; for example, the lowest point C is at angular position $\theta = 0$.](image2)
component of the centripetal acceleration. The average vertical component of the centripetal acceleration is
\[
\frac{1}{\pi/2} \int_{-\pi/2}^{\pi/2} a_c \cos \theta \, d\theta = \frac{a_c}{\pi/2} = a_c \frac{2}{\pi},
\]
where \(\theta\) is the angle from F to any given infinitesimal segment of the chain relative to the vertical direction, as indicated in Fig. 3. Assembling all the terms for the vertical component of Newton's second law for section BD gives
\[
2T_{BD} - \lambda \pi rg = \lambda \pi r \left( \frac{\frac{v^2}{2} + a}{4r} - \frac{a}{2} \right).
\] (4)

Again putting \(r \to 0\) results in
\[
2T_{BD} = \lambda \frac{v^2}{2} \Rightarrow v^2 = \frac{4T_{BD}}{\lambda}.
\] (5)

The left-hand equation predicts that \(T_{BD} = 0\) if \(\lambda = 0\). Thus in Hewitt's race between a free block and a block attached to the end of a folded chain, the chained block only wins if the chain has mass.\(^5\) Also \(T_{BD} = 0\) if \(v = 0\). That is, before the release of end E the tension in the chain at the bottom fold is zero, as expected for a static hanging chain.

Now consider section DE of the chain in isolation, as sketched in Fig. 4. Newton's second law for it becomes
\[
T_{BD} + \lambda g(L - y)/2 = \lambda a(L - y)/2.
\] (6)

Solve this equation for \(T_{BD}\) and substitute it into the right-hand side of Eq. (5) to get
\[
v^2 = 2(L - y)(a - g).
\] (7)

Separately using the chain rule, the acceleration is
\[
a = \frac{dv}{dt} = \frac{dy}{dt} \frac{dv}{dy} = v \frac{dv}{dy} = \frac{d}{dy} \left( \frac{v^2}{2} \right),
\] (8)

so that Eq. (7) becomes
\[
(L - y) \frac{d^2v}{dy^2} - v^2 = 2g(L - y) \Rightarrow \frac{d}{dy} \left[ (L - y)v^2 \right] = 2g(L - y),
\] (9)

which can be integrated with respect to y as
\[
(L - y)v^2 = gvy(2L - y) \Rightarrow v(y) = \sqrt{\frac{gy(2L - y)}{L - y}},
\] (10)

after accounting for the initial condition \(v(0) = 0\). Thus
\[
\frac{v}{v_{free}} = \sqrt{\frac{1 - y/2L}{1 - y/L}},
\] (11)

which is indeed greater than 1 for any \(y\) in the range \(0 < y < L\), as plotted in Fig. 5. For example, the speed of the free end E of the chain is double the free fall speed when \(y = 6L/7\). The speed diverges when \(y = L\) as the last bit of the chain “whips” around the bottom turn.\(^4\) Likewise the acceleration \(a\), which can be found by substituting Eq. (10) into (7), diverges when \(y = L\). This divergence in \(v\) and \(a\) does not occur for a real chain because the last link at the free end has a finite length and mass (rather than continuously decreasing to zero) analogous to the “cracker” at the tip of a whip.\(^25\)

Equation (10) is consistent with conservation of mechanical energy,\(^26\) in which the kinetic energy lost by a link in moving from point D to B in Fig. 1 is added to the kinetic energy of section DE of the chain. As the link turns the corner at point C, it is pulled back by section DE, thereby slowing the link down to rest and correspondingly speeding up the falling section of chain. Think of this bottom link as a small grappling hook connected to section DE by a very stiff elastic band. To go around the corner, that band must stretch slightly, thereby slowing down the hook. Just as the hook comes to rest, it attaches without dissipation to fixed point B. The elastic potential energy in the band now pulls on and speeds up the rest of the falling chain, giving it extra kinetic energy.

Acknowledgments

Seth Rittenhouse and later Alpha Wilson proposed that section BD of the chain behaves as if it rotates around a falling pulley. Bob Morse suggested analyzing the torques on that section. Rich Downey commented on the tension along the chain.

References


7. The appendix is available at TPT Online, http://dx.doi.org/10.1119/1.5033873, under the Supplemental tab.


11. When points A and O in Fig. 1 are far apart, sections AB and DE will initially be exact catenaries [as in F. Behroozi, “In praise of the catenary,” *Phys. Teach.* 56, 214–217 (April 2018)] only for a uniform string rather than for a chain having discrete links.


13. The tangential acceleration of the entire chain in the co-moving frame is $a/2$, which is the average in the lab frame of the acceleration $a$ of section DE and the zero acceleration of section AB.

14. "Chain Drop Answer 2," https://www.youtube.com/watch?v=X-QFAB0gEtE.


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Online Appendix for “Newtonian Analysis of a Folded Chain Drop”

A novel rocket equation approach was used by de Sousa and Rodrigues\(^1\) to incorrectly deduce that the released end of the chain is in freefall such that its acceleration is \(a = g\) . Surprisingly, the specific flaw in the analysis is not pinpointed in a subsequent related paper\(^2\) by one of the authors. Equation (14) of Ref. 1 can be rewritten as

\[
\frac{d}{dt}(M\dot{v}) = Mg + u \frac{dM}{dt}
\]

where they put \(u = \dot{v}\). Here \(M\) is the mass of section \(DE\) of the chain in Fig. 1 in the main body of the present article which is equal to \(\lambda (L - y)/2\). The problem is that de Sousa and Rodrigues did not account for the semicircular bend in the chain, which is effectively a third subsystem (in addition to sections \(AB\) and \(DE\) of the chain). Applying their Eq. (11) with subsystem I taken to be section \(BD\) of the chain (moving downward with average speed \(u = \dot{v}/2\)) and subsystem II taken to be section \(DE\) of the chain, then Eq. (A1) gives the correct answer. That is, Eq. (A1) is correct only for \(u = \dot{v}/2\) rather than \(u = \dot{v}\), as also noted in Sec. III of Wong and Yasui.\(^3\) As a check, Eq. (11) of Ref. 1 can be used again, with the same subsystem I but this time with subsystem II taken to be section \(AB\) of the chain. Then Eq. (A1) becomes

\[
\frac{d}{dt}(0) = M'\dot{g} - T_A + u \frac{dM'}{dt}
\]

where \(M' = \lambda (L + y)/2\) is the mass of section \(AB\) of the chain which is at rest, explaining the zero momentum on the left-hand side of Eq. (A2). Here \(T_A\) is the tension with which the support (indicated by hash marks in Fig. 1 in the main body of the present article) pulls upward on end \(A\) of the chain, equivalent to what de Sousa and Rodrigues call \(-T(x)\). Equation (A2) can be rearranged to find

\[
T_A = \frac{1}{2} \lambda (L + y)g + \frac{1}{4} \lambda \dot{v}^2
\]

in agreement with Eq. (13) of Calkin and March,\(^4\) in contrast to the erroneous Eq. (16) of de Sousa and Rodrigues.\(^1\)

An alternative correct approach is to consider all of the chain from points \(B\) to \(E\) in Fig. 1 (in the main body of the present article) to be one system of mass \(M\) and downward speed \(\dot{v}\). Since the two forces on this system are \(Mg\) downward and tension \(T_{BD}\) upward on end \(B\), one finds

\[
\frac{d}{dt}(M\dot{v}) = Mg - T_{BD}.
\]

Likewise consider a second system to be section \(AB\) of mass \(M'\) at rest. The three forces on this second system are \(M'\dot{g}\) downward, \(T_A\) upward on end \(A\), and \(T_{BD}\) downward on end \(B\) so that

\[
\frac{d}{dt}(0) = M'\dot{g} - T_A + T_{BD}.
\]
The reader is invited to check that Eqs. (A4) and (A5) are fully consistent with the equations presented in the main body of the present article. In particular, the sum of these two equations implies that the momentum change of the whole chain equals the net external force acting on it.

Methods similar to those presented in this Appendix can be used to find the tension at any other point of interest along sections AB or DE of the chain as a function of the distance y that free end E has fallen (in Fig. 1 in the main body of the present article).