Analytic solution for a variable-mass snowball

In the March 2019 issue, Scott Rubin analyzes the motion of a snowball rolling without slipping down a snowy inclined plane (making angle $\theta$ with the horizontal) and accreting additional snow as it does so.\(^1\) Equations (12) and (13) of Rubin's paper are two coupled differential equations for $v$ and $r$, but they cannot be solved analytically in terms of elementary functions of independent variable $t$. Instead, Rubin performs a numerical integration in a spreadsheet to find $v(t)$, along with the translational acceleration $a(t)$ of the snowball's center of mass.

However, it is possible to find $v$ and $a$ analytically as a function of a different independent variable, namely the snowball's radius $r$. This result is arguably even more useful than knowing how $v$ and $a$ vary with time, because we can directly relate $r$ to the distance $s$ the snowball has rolled down the slope as

$$s = \int r \, d\phi = \int r \, \frac{d\phi}{dr} \, dr = \int r \, \frac{2\pi}{k} \, dr = \frac{\pi}{k} (r^2 - r_0^2)$$  \hspace{1cm}(1)$$

using $\frac{dr}{d\phi} = k / 2\pi \phi$ where $\phi$ is the angle through which the ball has rolled and $r_0$ is the snowball's initial radius. Find $v(r)$ as follows. Solve the right-hand part of Rubin's Eq. (12) for $v$ and substitute it into his Eq. (13) to obtain

$$C - 60V^2 = 14rA$$  \hspace{1cm}(2)$$

after simplifying, where $V \equiv dr/dt$ and $A \equiv dV/dt$, and where $C = 5\pi^{-1}gk \sin \theta$ is a constant. Now observe that

$$A = \frac{dr}{dt} \frac{dV}{dr} = V \frac{dV}{dr},$$ \hspace{1cm}(3)$$

which is the same trick one uses to prove the work-energy theorem. Substitution of Eq. (3) into (2) results in an equation whose remaining variables $r$ and $V$ can be separated into

$$\int_0^r \frac{dV}{CV^{-1} - 60V} = \int_0^r \frac{dr}{14r}$$  \hspace{1cm}(4)$$

where the snowball initially has speed $v = 0$ and hence $V = dr/dt$ must also be initially zero.

After integrating, this result becomes

$$V(r) = \sqrt{\frac{gk \sin \theta}{12\pi} \left[ 1 - \left( \frac{r_0}{r} \right)^{\frac{60}{7}} \right]},$$  \hspace{1cm}(5)$$

yielding

$$v(r) = \frac{\pi g r_0^2 \sin \theta}{3k} \left( \frac{r}{r_0} \right)^{\frac{60}{7}} - \left( \frac{r_0}{r} \right)^{\frac{60}{7}}.$$  \hspace{1cm}(6)$$

Finally, $a$ can be computed as

$$a(r) = V \frac{dv}{dr} = \frac{g \sin \theta}{6} \left[ 1 + 23 \left( \frac{r_0}{r} \right)^{\frac{60}{7}} \right],$$  \hspace{1cm}(7)$$

which has an initial value of $(5/7) \, g \sin \theta$ and a terminal value of $(1/6) \, g \sin \theta$. Equations (6) and (7) are plotted in Fig. 1 and have a similar shape to Rubin's graphs of $v(t)$ and $a(t)$ because $V = dr/dt$ rapidly approaches a constant value of $(12\pi)^{-1/2}$ $(gk \sin \theta)^{1/2}$ so that $r$ can then be linearly rescaled as $t$.


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Fig. 1. Graphs of the acceleration and velocity of the snowball as a function of the distance it has rolled downhill from Eqs. (1), (6), and (7) using Rubin's values of \( k = 5 \text{ cm} \), \( \theta = 30^\circ \), and \( r_0 = 2 \text{ m} \).
Derivation of Four Equations in the Snowball Letter

Equation (2)

The last equality in Eq. (12) of Rubin rearranges into

\[ v = \frac{2\pi r}{k} V \]  \hspace{1cm} (A1)

where \( V = \frac{dr}{dt} \). Substituting Eq. (A1) into Eq. (13) of Rubin leads to

\[ g \sin \theta - \frac{23k}{10\pi} \left( \frac{2\pi rV}{k} \right)^2 = \frac{72\pi}{5} \frac{d(rV)}{dt} \]  \hspace{1cm} (A2)

which simplifies to

\[ g \sin \theta - \frac{46\pi}{5k} V^2 = \frac{14\pi}{5k} \left( V^2 + rA \right) \Rightarrow g \sin \theta - \frac{60\pi}{5k} V^2 = \frac{14\pi}{5k} rA \]  \hspace{1cm} (A3)

where \( A = \frac{dV}{dt} \). Multiplying through by \( \frac{5k}{\pi} \) turns Eq. (A3) into Eq. (2) in the letter.

Equation (5)

As one can verify by differentiation, the two integrals in Eq. (4) of the letter are

\[ -\frac{\ln(C - 60V^2)}{120} \bigg|_0^V = \frac{\ln(r)}{14} \bigg|_{r_0}^r \Rightarrow \ln \left( \frac{C}{C - 60V^2} \right)^{1/120} = \ln \left( \frac{r}{r_0} \right)^{1/14} \]  \hspace{1cm} (A4)

Take the antilogarithm and then the 120th power of both sides to obtain

\[ \frac{1}{1 - 60V^2 / C} = \left( \frac{r}{r_0} \right)^{60/7} \]  \hspace{1cm} (A5)

Finally take the reciprocal of both sides, substitute in \( C = 5\pi^{-1} gk \sin \theta \), and then solve for \( V \) to get Eq. (5) in the letter.

Equation (6)

Substitute Eq. (5) from the letter into Eq. (A1) above to obtain

\[ v = \sqrt{\frac{2\pi r_0}{k} \frac{r}{r_0} \left( \frac{gk \sin \theta}{12\pi} \right) \left[ 1 - \left( \frac{r_0}{r} \right)^{60/7} \right]} \]  \hspace{1cm} (A6)

which simplifies to become Eq. (6) in the letter.
Equation (7)

The acceleration of the snowball is

\[
a = \frac{dr}{dt} \frac{dv}{dr} = V \frac{dV}{dr} = \sqrt{\frac{g k \sin \theta}{12 \pi} \left[ 1 - \left( \frac{r_0}{r} \right)^{60/7} \right] \frac{\pi g r_0^2 \sin \theta}{3k} - \frac{2r}{r_0^2} + \frac{46r_0^{46/7}}{7r^{53/7}}}
\]

using Eqs. (5) and (6) in the letter. This result simplifies to become Eq. (7) in the letter.
A Variable-Mass Snowball Rolling Down a Snowy Slope

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The stereotypical situation of a snowball picking up both mass and speed as it rolls without slipping down a hill provides an opportunity to explore the general form of both translational and rotational versions of Newton’s second law through multivariable differential equations. With a few reasonable assumptions, it can be shown that the snowball reaches a terminal acceleration. While the model may not be completely physically accurate, the exercise and the resulting equation are useful and accessible to students in a second year physics course, arguably.

First, consider the free-body diagram of a spherical snowball on an incline with angle \( \theta \) (Fig. 1). Here, \( F_g \) is the downward gravitational force, \( F_N \) is the normal force of the inclined plane on the snowball, \( F_f \) is the force of friction, \( m \) is the mass of the snowball, and \( g \) is the acceleration of gravity.

The net external force acting on the snowball is then:

\[ F_{\text{net}} = mg \sin \theta - F_f. \]  (1)

For a thorough understanding of the situation, it is important to consider both the translational and the rotational aspects of the snowball’s motion. For the translational motion, use the general form of Newton’s second law:

\[ F_{\text{net}} = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}. \]  (2)

And therefore,

\[ mg \sin \theta - F_f = m\frac{dv}{dt} + v\frac{dm}{dt}. \]  (3)

Note that Eq. (2) is only valid for a well-defined system upon which the net force is applied.1–3 In this case, the system includes both the snowball and all the resting snowflakes waiting to be accreted. Since these waiting snowflakes remain at rest until they become part of the snowball, their velocities and accelerations vanish and Eq. (2) should be valid.

Now consider the rotational aspect of the snowball’s motion. The force of friction, \( F_f \), causes the torque acting on the snowball and is necessary for rotation to occur. Using the definition of torque \( (\tau = rF) \), where \( r \) is the radius of the snowball, we have

\[ mg \sin \theta - \frac{\tau}{r} = m\frac{dv}{dt} + v\frac{dm}{dt}. \]  (4)

Since the radius, mass, and velocity are all changing, we need to use the general version of the rotational form of Newton’s second law:

\[ \tau = \frac{dL}{dt} = I \frac{d\omega}{dt}, \]

where \( L \) is angular momentum, \( I \) is rotational inertia, and \( \omega \) is the angular velocity:

\[ \tau = \frac{dL}{dt} = I \frac{d\omega}{dt} + \omega \frac{dI}{dt}. \]  (5)

Equation (4) becomes

\[ mg \sin \theta - \frac{I}{r} \frac{d\omega}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}. \]  (6)

Next, substitute \( I = \frac{2}{5}mr^2 \) for a uniformly dense sphere:

\[ mgr \sin \theta - \frac{2}{5}mr^2 \frac{d\omega}{dt} - \frac{2}{5}m \frac{v^2}{t} \frac{dm}{dt} - \frac{4}{5}mr \frac{dr}{dt} = \frac{mr}{t} \frac{dv}{dt} + vr\frac{dm}{dt}. \]  (7)

Combining like terms and applying the relationship \( \omega = \frac{v}{r} \) yields

\[ mgr \sin \theta - \frac{7}{5}vr \frac{dm}{dt} - \frac{2}{5}m \frac{v^2}{t} \frac{dm}{dt} = \frac{7}{5}mr \frac{dv}{dt}. \]  (8)

But the rates of change of mass and radius are related. For a uniformly dense sphere of density \( \rho \),

\[ m = \frac{4}{3} \pi r^3 \rho, \]  (9)

which, upon differentiation yields

\[ dm = 4\pi r^2 \rho dr = \frac{3m}{r} \frac{dr}{dt}. \]  (10)

Substituting this relationship into Eq. (8) yields

\[ gr \sin \theta - \frac{23}{5}v \frac{dr}{dt} = \frac{7}{5}r \frac{dv}{dt}. \]  (11)

A reasonable assumption is the snowball’s radius increases by a constant value \( k \) in every rotation:

\[ \frac{\Delta r}{\Delta t} = \frac{k}{2\pi / \omega} = \frac{kv}{2\pi r} \approx \frac{dr}{dt}. \]  (12)

That gives a differential equation of motion:

\[ g \sin \theta - \frac{23v^2 k}{10\pi r^2} = \frac{7}{5} \frac{dv}{dt}. \]  (13)

This isn’t such an easy equation to solve since the radius is also changing, but one more time derivative is instructive. Note that the term \( g \sin \theta \) is constant. If \( a = \frac{dv}{dt} \), we have

\[ \frac{23k \left( 2va \right)}{10\pi \left( r^2 - \frac{v^3}{\pi \rho} \right)} = \frac{7}{5} \frac{da}{dt}. \]  (14)
This suggests a terminal acceleration \( a_T \) when \( a = \frac{v^2 k}{2 \pi r^2} \), which, with substitution into Eq. (13), yields

\[
a_T = \frac{1}{6} g \sin \theta. \tag{15}\]

This is a satisfying result for two reasons. First of all, it is relatable. Any student who has internalized the concept of terminal velocity should be able to extend their understanding to terminal acceleration. Secondly, it is intuitive. It wouldn’t make sense for a snowball to get slower as it rolled down a hill. Neither would it make sense for the snowball to accelerate as much as the usual value of a uniformly dense sphere of constant radius rolling without slipping down an incline, namely:

\[
\left[ \frac{g \sin \theta}{\sqrt{5/7}} \right].
\]

It also makes sense that the value \( k \), the rate the radius increases per rotation, doesn’t affect the final acceleration. It only affects the number of rotations before the terminal acceleration is reached.

It is fair to note that the edges of the snowball may not gain mass at the same rate as the point of contact at its center, and hence the relationship \( \frac{dr}{dt} \approx \frac{kv}{2 \pi r} \) may be a poor approximation. However, any model that maintains a spherical shape would require a similar approximation. Nevertheless, it is instructive to apply this analysis to a rolling “snow-log”—a uniformly dense cylinder—to which the approximation more clearly applies.

Equation (8), for example, can be generalized to accommodate any rotating rigid body with a rotational inertia of \( I = \frac{\lambda m r^2}{2} \), where \( \lambda \) is a dimensionless constant between 0 and 1:

\[
mg \sin \theta - \frac{1}{(1 + \lambda) \nu r} \frac{dP}{dt} - \lambda mv \frac{dv}{dt} = \frac{1}{(1 + \lambda) \nu r} (\frac{dI}{dt}). \tag{16}\]

For a uniformly dense cylinder with length \( L \) and density \( \rho \), \( m = \pi r^2 L \rho \) and \( dm = 2 \pi r \frac{dx}{dt} \). Since \( \lambda \) for a uniformly dense cylinder is 1/2, Eq. (13) becomes

\[
g \sin \theta - \frac{7k v^2}{4 \pi r^2} = \frac{1}{2} \frac{dv}{dt},
\]

which yields a terminal acceleration of

\[
a_T = \frac{1}{2} g \sin \theta. \tag{17}\]

These results are reminiscent of the classic problem in which a drop of rain that gains mass as it falls from rest through a mist accelerates at a constant rate of \( g/2 \), \( g/4 \), and \( g/7 \) if increasing mass is proportional to time, speed, and number of droplets “swept out,” respectively.\(^4\text{-}\text{6}\)

While a snowball, unlike a raindrop, doesn’t “sweep out” stationary snowflakes, and as it gets faster and larger its mass clearly increases more quickly than a simple proportionality with time, it is instructive to follow the approach of Dick\(^7\) and analyze Eq. (13) numerically. In the numerical model, I set initial values for \( k \) (in meters), initial radius (in meters), and initial velocity (in m/s). I used time steps of 0.01 in an incremental version of Eqs. (12) and (13):

\[
\frac{7}{5} \Delta v = \left[ g \sin \theta - \frac{23 v^2 k}{10 \pi r^2} \right] \Delta t \tag{18}\]

\[
\Delta r = \left( \frac{k v}{2 \pi r} \right) \Delta t. \tag{19}\]

Increasing \( k \) or decreasing \( r \) reduced the time it took to reach terminal acceleration, although, as indicated above, the terminal acceleration remained the same for any size snowball on a particular inclined plane.

The spreadsheet is available on request, but Figs. 2 and 3, respectively, show speed and acceleration for a representative snowball. Under these conditions, the snowball is very close to terminal acceleration by about 20 seconds.

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References