Rocket Propulsion, Classical Relativity, and the Oberth Effect

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The role of work and mechanical energy in classical relativity has been a subject of renewed interest in this publication. Here we present a problem that illustrates the relationship between impulse and kinetic energy for a rocket-powered object that can also change its gravitational potential energy. The same introductory physics principles lead to a remarkable result when applied to the mechanics of spacecraft—the Oberth effect—whereby a small impulse can cause a large change in a rocket’s orbital energy without violating any conservation laws.

Problem statement

Refer to Fig. 1. A rocket-powered skateboard (hereafter the “rocket”) is about to roll downhill starting from rest at A. At what position (A, B, or C) should the rocket fire a brief impulse to maximize its final speed on the other side of the valley, or does it not matter? To simplify this problem, assume that $g$ is constant, dissipative forces such as friction and air drag are negligible, and all speeds in the planet’s reference frame are such that classical (Galilean) relativity applies.

With no burst from the rocket engine, the rocket will roll back and forth in the potential well between points A and C. Astute students will recognize that firing the burst at A or C has the same effect on the rocket, since it is at rest at the same height for both locations. But they may be unsure about the effect of firing it at B, where the rocket is moving.

In the following analysis, set the zero level of gravitational potential energy at the height of the right side of the valley. The rocket starts on the left at a depth $d_A$ below this level, and the valley floor at B has a depth $d_B > d_A$ (as indicated in Fig. 1). The rocket ejects an exhaust mass $m_{ex}$ backwards in a brief burst of duration $\delta t$. This expelled fuel moves at speed $v_{ex}$ relative to its initial velocity (which is also the velocity of the rocket + fuel system’s center of mass). Let the remaining mass of the rocket be $M_R$ and its change in speed $\Delta v$. From momentum conservation for the rocket and expelled fuel, valid in any inertial frame,

$$M_R \Delta v = m_{ex} v_{ex} = F \delta t \tag{1}$$

for a constant force $F$ between the rocket and the exhaust. For a time-varying force, the impulse vector’s magnitude given by the right-hand side of Eq. (1) has to be replaced by an integral, but that does not affect the analysis.

Short solution: Same impulse but not the same work

From Eq. (1), the impulse causes the same $\Delta v$ regardless of the rocket’s motion. However, if the rocket is moving in the planet’s frame at speed $v$ just prior to the impulse, the resulting change in its kinetic energy in that frame is

$$\Delta K_R = \frac{1}{2} M_R (v + \Delta v)^2 - \frac{1}{2} M_R v^2 = \frac{1}{2} M_R (\Delta v)^2 + M_R v \Delta v, \tag{2}$$

which increases with $v$. Thus the rocket gains more kinetic energy for the same impulse if it is already moving, and therefore the impulse should be applied when the rocket is moving fastest, at the valley bottom B in Fig. 1.

The effect of the impulse on the rocket’s final speed is derived below. But why does the same $\Delta v$ cause a larger $\Delta K_R$ when applied at a nonzero speed? The short answer is that the work done on the rocket by the force, $F \delta x$ measured in the planet frame, is greatest at B, since the moving rocket covers more distance $\delta x$ there for the same time interval $\delta t$. This work is equal to the rocket’s change in kinetic energy $\Delta K_R$ when both are calculated in the planet’s reference frame.\(^2,4\)

Extra kinetic energy for the same fuel expended?

In the planet’s frame, giving the rocket an impulse when it is already moving appears to provide it with “extra” kinetic energy, compared to firing the impulse when the rocket is at rest. That seems paradoxical. An observer moving with the rocket always measures the same chemical energy converted, to expel the same fuel mass at the same relative exhaust velocity. So where does the rocket’s additional kinetic energy come from?

The resolution to this paradox requires accounting for the change in the kinetic energy of the exhaust mass expelled. In the planet frame, if the rocket and fuel are already moving forwards at speed $v$, as shown in Fig. 2, the exhaust speed relative to the planet is $|v_{ex} - v|$. (We take the absolute value here to include the case $v > v_{ex}$ for which the expelled fuel will then move in the same direction as the rocket.) The change in the kinetic energy of the expelled mass is

$$\Delta K_{ex} = \frac{1}{2} m_{ex} (v_{ex} - v)^2 - \frac{1}{2} m_{ex} v^2 = \frac{1}{2} m_{ex} v_{ex}^2 - m_{ex} v^2, \tag{3}$$

DOI: 10.1119/1.5126818
which can be positive, negative, or zero. Adding Eqs. (2) and (3), and using Eq. (1), the total change in kinetic energy of the rocket + fuel is

$$\Delta K_{\text{total}} = \frac{1}{2} m_{\text{ex}} v_{\text{ex}}^2 + \frac{1}{2} M_R (\Delta v)^2 \equiv \Delta E_{\text{chem}}, \quad (4)$$

which is independent of $v$. Here $\Delta E_{\text{chem}}$ is the kinetic energy gained from the conversion of the fuel's chemical energy via the engine, a quantity for which observers in all inertial frames must agree.

In the center of mass frame, $\Delta E_{\text{chem}}$ is shared by the rocket and exhaust mass according to Eqs. (1) and (4). In the planet frame, from Eqs. (3) and (4), for rocket speeds $v > v_{\text{ex}}/2$ we find $\Delta K_{\text{ex}} < 0$ and hence $\Delta K_R > \Delta E_{\text{chem}}$, i.e., the rocket gains more kinetic energy than that provided by combusting the fuel! In that case, the additional energy gained by the rocket is obtained from a portion of the fuel's mechanical energy.

**Calculating the final speed**

If the rocket rolls down to point B and fires its impulse there, what will be its final speed when it emerges on the right side of the valley?

The mechanical energy of a system is the sum of its kinetic and potential energies, and is constant in the absence of external work. In the planet frame before the rocket fires, the initial mechanical energy of the rocket+fuel $E_{\text{total,i}}$ equals their gravitational potential energy at position A in Fig. 1, since they are stationary there,

$$E_{\text{total,i}} = -(M_R + m_{\text{ex}})gd_A. \quad (5)$$

The speed $v_B$ at the valley bottom B, just before firing the burst, is found from mechanical energy conservation as

$$E_{\text{total,i}} = \frac{1}{2} (M_R + m_{\text{ex}}) v_B^2 -(M_R + m_{\text{ex}})gd_B = -(M_R + m_{\text{ex}})gd_A$$

$$\Rightarrow v_B = \sqrt{2g(d_B - d_A)}. \quad (6)$$

Just after firing the impulse at B, the rocket and expelled fuel have speeds of $v_B + \Delta v$ and $|v_B - v_{\text{ex}}|$, respectively (cf. Fig. 2). Using Eqs. (2) and (3), with $v = v_B$ from Eq. (6), their respective final mechanical energies $E_{R,f}$ and $E_{\text{ex},f}$ become

$$E_{R,f} = \frac{1}{2} M_R (v_B + \Delta v)^2 - M_R gd_B$$

$$= \frac{1}{2} M_R (\Delta v)^2 - M_R gd_A + M_R \Delta v \sqrt{2g(d_B - d_A)} \quad (7)$$

and

$$E_{\text{ex},f} = \frac{1}{2} m_{\text{ex}} (v_B - v_{\text{ex}})^2 - m_{\text{ex}} gd_B$$

$$= \frac{1}{2} m_{\text{ex}} v_{\text{ex}}^2 - m_{\text{ex}} gd_A - m_{\text{ex}} v_{\text{ex}} \sqrt{2g(d_B - d_A)}. \quad (8)$$

Compared to the final mechanical energies if the impulse were fired at A, Eqs. (7) and (8) have an additional term on the right-hand side. (As a check, set $d_B = d_A$, as if the valley bottom did not exist, and the third term in each equation vanishes.) From Eq. (1), those terms are equal and opposite, so that the sum of $E_{R,f}$ and $E_{\text{ex},f}$ gives a total final energy for the [rocket+fuel] system

$$E_{\text{total,f}} = -(m_{\text{ex}} + m_{\text{ex}})gd_A + \frac{1}{2} m_{\text{ex}} v_{\text{ex}}^2 + \frac{1}{2} M_R (\Delta v)^2$$

$$= E_{\text{total,i}} + \Delta E_{\text{chem}}, \quad (9)$$

as expected from Eq. (4).

After the impulse at the valley bottom, use conservation of mechanical energy for the rocket alone to find its final speed $v_{R,f}$ on reaching the top of the valley on the right, where the gravitational potential energy is zero. From Eq. (7),

$$\Rightarrow v_{R,f} = (\Delta v)^2 - 2gd_A + 2\Delta v \sqrt{2g(d_B - d_A)} \quad (10)$$

The final speed when the impulse is fired instead at A or C can be found from Eq. (10) by setting $d_B = d_A$.

**Rocket fuel carries chemical and mechanical energy**

Equations (7) and (10) provide a remarkable result. If we increase the valley depth $d_B$, the rocket's final mechanical energy, and hence its final speed, can be as large as one wants! If $d_B$ is deep enough, it is even possible for a rocket to start at A and escape on the right with $v_{R,f} > \Delta v$, eventually passing an identical rocket that starts from rest on the valley's right side and fires its burst there.
If one considers the energetics of the rocket only, this result seems surprising. However, the total post-impulse mechanical energy of the rocket plus fuel is conserved, and the exhaust mass expelled at B ends up with less mechanical energy as \( d_E \) is increased. For example, from Eq. (8) the ejected fuel mass will become trapped in the potential well between A and C if \( E_{cex} < -m_e g d_A \).

Examining the right-hand side of Eq. (7), we see that the rocket always gains energy from some of the fuel’s converted chemical energy, given by the first term (cf. Eq. 4). By firing its burst at the lowest point B, the rocket can also “steal” additional energy, given by the last term, from the original mechanical energy carried by the fuel. (The remaining middle term equals the initial mechanical energy of the rocket body.)

The Oberth effect

The application of this result to spaceflight was first suggested in 1929 by astrodynamics pioneer Hermann Oberth. In a bound elliptical orbit around a central body, the fuel carries more mechanical energy than the same mass at rest on the body’s surface. As with our ground-based example, the greatest boost to the rocket’s orbital energy is produced by a prograde (forward-acting) impulse made near periapsis, the lowest and fastest point in its path, indicated as position B in Fig. 3 by analogy with Fig. 1.

The Oberth effect refers to this increase in the rocket’s orbital energy, which will either increase the size of its elliptical orbit, or its hyperbolic excess speed as shown in Fig. 3.9 Conversely, retro-firing at periapsis is the most effective way to reduce the spacecraft’s mechanical energy (by transferring a large portion of it to the expelled fuel). Such a maneuver is often used to achieve orbital capture, as exemplified by the Juno mission’s insertion into orbit around Jupiter.

Since orbital motion occurs in a 1/r gravitational potential, with angular momentum as an additional constraint, the details of the orbital Oberth effect differ from the ground-based problem analyzed here, but the physical principles are the same. We invite users of orbital mechanics software such as Orbiter, STK, or Kerbal Space Program to simulate the effects of an impulse on a spacecraft at different locations in its elliptical orbit. A computational physics simulation of Fig. 1 (e.g. in Algodoo), or a laboratory apparatus using a spring-powered mass launcher running on a flexible track, would also serve as a demonstration of these principles. Students should appreciate that Newtonian conservation laws apply to rocket propulsion for all inertial observers, as long as mechanical energy changes in the rocket and exhaust are taken into account.

Additional PowerPoint slides for presenting this problem to students, and an STK animation of Fig. 3, are provided in an online supplement. Thanks to Devin Potratz for assistance in producing Fig. 1.

References
1. E. Ginsburg, “Simple examples of the interpretation of changes in kinetic and potential energy under Galilean transforma-
5. We desire both the largest possible initial speed \( v \) and exhaust speed \( v_{ex} \) (also referred to as the fuel’s specific impulse in \( \text{m/s} / \text{kg} \)) in order to provide the largest boost to the rocket, according to Eqs. (1) and (2). Also see M. Iona, “Efficiency in rocket propulsion,” Phys. Teach. 16, 600 (Dec. 1978).
6. The gravitational potential energy of the rocket and fuel should properly be assigned to the systems of planet + rocket and planet + fuel, respectively. For brevity, we refer to the mechanical energy of the “rocket” or “fuel” alone. If we assume a near-infinite mass for the planet, we can neglect any changes in its motion as a result of its interactions with the rocket or fuel.
9. Perhaps easier to visualize (by reversing the arrows in Fig. 3), the time-reverse version of this maneuver is an inelastic head-on collision between an orbiting fuel mass and an incoming rocket, whereby the combined object ends up with a mechanical energy less than that of the colliding objects by an amount \( \Delta E_{coll} \).
10. “Inserting Juno into orbit around Jupiter,” online at physicsfromplanetearth.wordpress.com/2016/07/10/.
11. The Oberth effect is distinct from a gravity-assist “slingshot” maneuver, which describes an elastic momentum and energy exchange between an unpowered spacecraft and a planet when both are under the gravitational influence of a third body, typically the Sun. See A. Bartlett and C. Hord, “The slingshot effect: Explanation and analogies,” Phys. Teach. 23, 466–473 (Nov. 1985). In the Oberth effect, no third body is required, and since the firing of the impulse does not change the motion of the rocket+fuel’s center of mass, it has no effect on the planet.
13. To obtain free educational licenses for Systems Tool Kit (STK), visit http://www.agi.com/education .
16. See the online supplements at TPT Online, http://dx.doi.org/10.1119/1.5126818 under the Supplemental tab.

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Carl Mungan enjoys teaching across the entire undergraduate physics curriculum, and thinks using clicker questions such as the one in Fig. 1 is beneficial for undergraduates of all levels, and not just introductory students. mungan@usna.gov
Supplement to “Rocket propulsion, classical relativity, and the Oberth Effect”, published in *The Physics Teacher*, 2019

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**Contents:**

Figure 1 (without annotations)
Alternate Figure 1
Additional figures
Animations with Systems Tool kit (separate PPT File):
  - Impulse fired at periapsis (cf. Figure 3)
  - Impulse fired at apoapsis for comparison
This rocket-powered skateboard is about to roll downhill. To escape on the right with the maximum speed, where should it fire a brief impulse? (Or does it not matter?)

Thanks to student Devin Potratz for assistance with the graphics.
This rocket-powered skateboard is about to roll downhill. To escape on the right with the maximum speed, where should it fire a brief impulse? (Or does it not matter?)

Thanks to student Devin Potratz for assistance with the graphics.
You get more $\Delta K$ for a fixed $\Delta v$ if you are already moving fast!

Explanation:

Impulse $F \delta t = m\Delta v$ is fixed, but Work $= F \delta x = F v \delta t$ increases with speed.

But where does this “extra” kinetic energy come from?
Where does this “extra” kinetic energy boost come from?

For the rocket:

\[
\Delta K_R = \frac{1}{2} M_R \left( v + \Delta v \right)^2 - \frac{1}{2} M_R v^2 = \frac{1}{2} M_R \Delta v^2 + v M_R \Delta v
\]

For the exhaust:

\[
\Delta K_{ex} = \frac{1}{2} m_{ex} \left( v - v_{ex} \right)^2 - \frac{1}{2} m_{ex} v_{ex}^2 = \frac{1}{2} m_{ex} v_{ex}^2 - v m_{ex} v_{ex}
\]

Since \( m_{ex} v_{ex} = M_R \Delta v \), when we add up the total \( \Delta K \),

\[
\Rightarrow \Delta K_{total} = \Delta K_R + \Delta K_{ex} = \frac{1}{2} M_R \Delta v^2 + \frac{1}{2} m_{ex} v_{ex}^2 = \Delta E_{chem} \quad - \text{invariant!}
\]

Classical Relativity ➔ invariance of impulse and total energy change

(Rocket converts same chemical ➔ kinetic energy independent of speed.)

For a rocket already moving fast, in our reference frame,

• more \( \Delta K_{total} \) goes to the rocket...and less to the exhaust, so

• rocket “steals” kinetic energy from exhaust!
Orbital Oberth effect (Figure 3)

Planet surface escape speed \( v_{SE} = \sqrt{\frac{2GM_P}{R_p}} \)

Set rocket mass \( M_R \equiv 16 \, m_{ex} \)
\( \rightarrow \Delta E_{chem} = \frac{17}{32} \, m_{ex} \, v_{ex}^2 \)

Set exhaust velocity \( v_{ex} \equiv 1.46286 \, v_{SE} \)
\( \rightarrow \Delta v = \frac{v_{ex}}{16} = 0.09143 \, v_{SE} \)
\( \rightarrow \Delta E_{chem} = 2.274 \, GMm_{ex}/R_p. \)

Initial rocket+fuel orbit (ellipse):
Set semi-major axis \( a \equiv 7.50 \, R_P \),
eccentricity \( e \equiv 0.8. \)
\( \rightarrow \) periapsis distance = \( 1.5 \, R_P \),
\( \rightarrow \) periapsis speed = \( 0.77460 \, v_{SE} \)
\( \rightarrow E_{total,i} = -\frac{17}{15}GMm_{ex}/R_p = -0.4984 \, \Delta E_{chem} \)

Resulting rocket orbit (hyperbola):
Semi-major axis \( a = -6.00 \, R_P \),
eccentricity \( e = 1.250. \)
\( \rightarrow E_{total,f} = +\frac{4}{3}GMm_{ex}/R_p \)
\( \rightarrow \Delta E_R = +2.4 \, GMm_{ex}/R_p = +1.0555 \, \Delta E_{chem}. \)

Resulting exhaust orbit (ellipse):
Semi-major axis \( a = 2.591 \, R_P \),
eccentricity \( e = 0.4211. \)
\( \rightarrow \Delta E_{ex} = -0.1263GMm_{ex}/R_p = -0.0555 \, \Delta E_{chem}. \)

Animation at https://vimeo.com/286053315