Sizing up Earth with a pair of sticks

In a recent Letter to the Editor,\(^1\) Hewitt asks whether measurements of the shadow angles cast by a pair of (radially) vertical sticks (which for accuracy could be tall flagpoles) that are a known distance \(s\) apart on a spherical astronomical body (which I will take to be Earth) can be used to find the circumference (or simply the radius \(R\)) of the planet if the sticks are not on a common line of longitude along with the Sun? The answer is yes, but only if both the horizontal and vertical shadow angles are measured, and not just the vertical angle (call it \(\alpha\)) as in Hewitt’s original “Figuring Physics” column.

Assume that a team of students have assembled at each flagpole. They have agreed to make simultaneous measurements at their locations on a sunny day (when the Sun is not directly overhead, so that the flagpoles cast shadows, as can be arranged even near the equator by not performing the project at noon). Coordinate axes with the origin at the center of Earth are established as follows. Define the +\(x\)-axis to point toward the Sun. Now consider a plane that includes that +\(x\)-axis and Earth’s arctic magnetic pole. Define the +\(z\)-axis to lie in that plane and to point toward Earth’s northern hemisphere, perpendicular to the +\(x\)-axis. Finally choose the +\(y\)-axis to point in the right-hand direction found by taking the cross product of unit vectors \(\hat{z}\) and \(\hat{x}\).

At each location, the students can first determine the vertical shadow angle \(\alpha\) of the flagpole as indicated in Fig. 1. If the pole has a known height \(H\), one method is to measure the shadow length \(L\) and compute
\[
\alpha = \tan^{-1} \left( \frac{L}{H} \right). \tag{1}
\]

Next assume each group has been provided with a magnetic compass and with the angular correction between the north reading on their compass and the point (call it \(P\)) that the +\(z\)-axis intersects the surface of Earth.\(^2\) After drawing a line starting from the base \(B\) of the flagpole and directed toward \(P\) (shown dashed in Fig. 2), the group can use a protractor to measure the horizontal angle \(\beta\) between the shadow and that dashed line.

To use \(\alpha\) and \(\beta\) to find the radius \(R\) of Earth,\(^3\) express them in terms of the colatitude \(\theta\) (measured downward from \(P\) along the dashed arc in Fig. 2) and longitude \(\phi\) (measured counter-clockwise around the \(z\)-axis starting from the \(x\)-axis as shown in Fig. 2) via
\[
\cos \alpha = \sin \theta \cos \phi \quad \text{and} \quad \tan \beta = \sec \theta \tan \phi \tag{2}
\]
as derived in the online appendix.\(^4\) (Readers are invited to check that these two equations give the expected answers for the four cases \(\phi = 0^\circ, \phi = 90^\circ, \theta = 0^\circ,\) and \(\theta = 90^\circ\).) Inverting Eq. (2) gives
\[
\tan \theta = \cot \alpha \sec \beta \quad \text{and} \quad \sin \phi = \sin \alpha \sin \beta, \tag{3}
\]
so that \(\theta\) and \(\phi\) can be deduced from the measured values of \(\alpha\) and \(\beta\). Finally, if the angular coordinates at the locations of the two flagpoles are \((\theta_1, \phi_1)\) and \((\theta_2, \phi_2)\) with the bases of the two flagpoles separated by arclength \(s\), then \(R = s/\gamma\), where\(^5\)
\[
\cos \gamma = \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2) + \cos \theta_1 \cos \theta_2. \tag{4}
\]

As a check, if \(\phi_1 = 0^\circ = \phi_2\) as in Hewitt’s special case when the two sticks lie along the line of longitude of the Sun, Eqs. (2) and (4) imply \(\gamma = \theta_1 - \theta_2 = \alpha_2 - \alpha_1 = \Delta \alpha\). For the values \(\Delta \alpha = 10^\circ\) and \(s = 200\) km in Hewitt’s column, one thereby deduces a planetary radius and circumference of
\[
R = \frac{200 \text{ km}}{10^\circ (2\pi \text{ rad}/360^\circ)} \Rightarrow 2\pi R = 7200 \text{ km} \tag{5}
\]
in agreement with his solution.

2. This angular correction depends both on the deviation between the axes of Earth’s magnetic and geographic poles, and on the Sun’s position on the celestial sphere at that particular time and date.
3. The same vertical shadow angle \(\alpha\) would be found for a flagpole whose base is located anywhere on a circle of radius \(QB\) surrounding point \(Q\) in Fig. 2. Thus \(\alpha\) alone does not give enough information to solve the general problem. One needs the horizontal bearing \(\beta\) to get a fix on the latitude and longitude of a pole.
4. See the Supplemental tab at TPT Online, http://dx.doi.org/10.1119/1.5131111, for the appendix.

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\(\text{letters}\)
Online Appendix for “Sizing up Earth with a pair of sticks”

A unit vector directed radially outward along a flagpole at colatitude $\theta$ and longitude $\phi$ is
\[
\hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\] (A1)

In addition, the solar ray in Fig. 1 is antiparallel to the unit vector $\hat{t} = (1, 0, 0)$. The dot product of these two vectors is
\[
\cos \alpha = \hat{r} \cdot \hat{t} = \sin \theta \cos \phi
\] (A2)
which reproduces the first equality in Eq. (2).

To continue, a unit vector $\hat{L}$ in the plane spanned by $\hat{r}$ and $\hat{t}$ but perpendicular to $\hat{r}$ (in the direction on the opposite side of the flagpole as the sun) would point in the direction of the shadow in Fig. 1. Thus
\[
\hat{L} = \frac{\sin \theta \cos \phi}{\sqrt{1 - \sin^2 \theta \cos^2 \phi}} \left( \sin \theta \cos \phi - \frac{1}{\sin \theta \cos \phi}, \sin \theta \sin \phi, \cos \theta \right)
\] (A3)
as can be verified by checking that (i) its dot product with $\hat{r}$ is zero, (ii) it is a linear combination of $\hat{r}$ and $\hat{t}$, (iii) it has unit magnitude, and (iv) it has a negative $x$ component.

Similarly, a unit vector $\hat{S}$ tangent to the dashed arc at the base of the flagpole in Fig. 2 and directed toward point P is
\[
\hat{S} = \cot \theta \left( -\sin \theta \cos \phi, -\sin \theta \sin \phi, \frac{1}{\cos \theta} - \cos \theta \right)
\] (A4)
as is verified by checking that (i) its dot product with $\hat{r}$ is zero, (ii) it is a linear combination of $\hat{r}$ and $\hat{z}$, (iii) it has unit magnitude, and (iv) it has a positive $z$ component. The dot product of Eqs. (A3) and (A4) is
\[
\cos \beta = \hat{L} \cdot \hat{S}
\] (A5)
which can be rearranged into the second equality in Eq. (2).

Taking the dot product of Eq. (A1) written twice—once with angles $\theta_1$ and $\phi_1$, and a second time with angles $\theta_2$ and $\phi_2$—gives $\cos \gamma$ in Eq. (4).
**Pair-of-Sticks Hypothesis**

Referring to the *Figuring Physics* “With Simply a Pair of Sticks” in the September 2018 issue of *TPT* (p. 361) and its answer in the October 2018 issue (p. 483), I hypothesized: **The size of any spherical body in overhead sunlight can be calculated by the shadows cast by a pair of vertical sticks a known distance apart in any direction.** A caveat in the *Figuring Physics* piece was that the shadow of one stick points to the other stick. This resulted in simple planar geometry, wherein the parallel shadow-casting sunbeams lie in the same plane as the two-sticks’ great circle. To calculate the circumference of the sphere, one only needs to know the angle of the vertex at the sphere’s center where vertical extensions of the sticks meet. Because of this coincidence of planes, the *difference in shadow angles* equals the vertex angle at the sphere’s center. (Or, if the overhead Sun is between the sticks and the shadows point away from them, the two shadow angles *add* to equal the vertex angle.) A full 360° divided by the angle of vertex directly leads to the circumference of the sphere. All is sweet and simple (which makes for a dandy project for pairs of students in distant cities—measuring shadows cast by their respective school flagpoles to calculate Earth’s circumference).

**My question to TPT readers:** What of the more general case where the shadows do not align with the arc distance between the pair of sticks? That is, when the planes of shadow-casting parallel sunbeams and the two-stick great circle do not coincide? Will spherical geometry, or something else, lead to the calculation of the sphere’s circumference? Or, is the hypothesis of “in any direction” invalid?

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