A Race Between Rolling and Sliding Up and Down an Incline

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Races between moving objects are an engaging way to teach dynamics to introductory physics students. One standard example consists in racing hollow and solid cylinders as they roll down an inclined plane. Another striking demonstration is a race between two marbles on side-by-side tracks that start and end together, but with one track taking a straight path and with the other following a cycloidal shape. In the present article, two objects (having the same mass and shape) are launched from the same starting height with the same speed up an inclined plane. One of the objects is subject to kinetic friction. The other object is not because it rolls without slipping on wheels along the incline with negligible rolling friction. Each object travels upward, slows down, and then turns around and retraces its path to its starting point. If the two objects are launched simultaneously, which one makes it back first?

![Free-body diagrams for a block of mass m sliding up and down a ramp inclined at angle θ to the horizontal.](image)

Although the object experiencing sliding friction travels slower (at any given nonzero altitude above the launch point) than does the rolling object at the same altitude, that does not guarantee that the rolling object will win the race, because it also travels to a higher maximum height than does the sliding object. In fact, it is shown analytically in this article that either object can win the race, depending on the angle of the incline. This prediction is then experimentally verified using carts and motion detectors of the sort found in introductory physics laboratories, which would make effective demonstrations or experiments for students in such courses.

### Theoretical prediction

As a simple model, consider two blocks, each of mass $m$, one of which is projected up along a rough portion of an inclined plane and the other along a frictionless portion beside it. The frictional force can be varied by changing the angle of the incline $θ$ for a fixed coefficient of kinetic friction $μ$. Air drag is neglected.

Free-body diagrams are sketched in Fig. 1 for a block sliding up and down a ramp. (The incline can either be rough with a nonzero value of $μ$ or frictionless with $μ = 0$, so that the following analysis applies to both cases.) Forces perpendicular to the surface of the ramp must balance so that the normal force is given by

$$N = mg \cos θ,$$

where $g = 9.8 \text{ N/kg}$ is the magnitude of Earth’s surface gravitational field. Thus the acceleration of the block when it is moving up the incline has the constant value

$$a_x = -g \sin θ - μg \cos θ$$

so that the time it takes the block to slide up the ramp to rest when given an initial speed $v_0$ is

$$t_{up} = \frac{v_0}{g \sin θ + μg \cos θ} = \frac{v_0}{g \sin θ (1 + μ / \tan θ)}.$$  (3)

The distance the block will move up along the incline during that time is

$$D = \frac{v_0^2}{2g \sin θ (1 + μ / \tan θ)}.$$  (4)

Assuming $μ < \tan θ$ (neglecting the difference between the static and kinetic coefficients of friction), the block will next slide back down the incline to its starting point. The acceleration during this descent will be

$$a_x = -g \sin θ + μg \cos θ$$

by inspection of the free-body diagram. The sign in front of $μ$ is flipped here compared to in Eq. (2), giving rise to a measurable change in the acceleration of a cart traveling up vs. down an incline. Constant-acceleration kinematics then predicts

$$t_{down} = \frac{\sqrt{2D}}{-a_x} = \frac{v_0}{g \sin θ \sqrt{1 - \mu^2 / \tan^2 θ}},$$

for the descent time using Eqs. (4) and (5). In the absence of friction, the time to go up or down the ramp is equal to

$$t_{1/2} = \frac{v_0}{g \sin θ}.$$  (7)

by setting $μ = 0$ in either Eq. (3) or (6). Thus the ratio of the total travel time with and without friction becomes

$$\frac{T_{frictional}}{T_{frictionless}} = \frac{t_{up} + t_{down}}{t_{1/2} + t_{1/2}} = \frac{1}{2} \left[ (1 + μ / \tan θ)^{-1} + (1 - μ^2 / \tan^2 θ)^{-1/2} \right].$$

(8)

Setting this ratio equal to 1, the result can be rearranged into a cubic equation whose unique real and nonzero root is

$$μ / \tan θ = 1 / \sqrt{2}.$$  (9)
as can be verified by substituting this value back into Eq. (8).

Consequently, this model predicts that a cart rolling up and down an incline will take the same amount of time to return to its launch point as will a cart whose wheels are locked (so that it slides on them with coefficient of friction $\mu$) if the angle of the track is chosen to have the critical value

$$\theta_c = \tan^{-1}\left(\sqrt{\frac{\mu}{2}}\right). \quad (10)$$

This angle satisfies the constraint that $\theta_c > \mu$, so the cart does not remain at rest at its turning point at the highest point it reaches above the floor.

**Experimental verification**

To test this prediction, rolling carts moving up and down rigid inclined tracks of the kind typically found in an introductory physics laboratory are used. The carts are PAScars (PASCO model ME-6950) after removing unneeded components to reduce their weight as much as possible.\(^5\) Spring-loaded launchers are available to project the carts up the incline with a reproducible initial speed.\(^6\) Motion detectors measure the positions of the carts along the tracks as a function of time.

The overall arrangement is photographed in Fig. 2, and a more detailed view of the cart launching mechanism is in Fig. 3. The zero position for a cart is found using the motion detector when the cart is stationary in its equilibrium position with the spring resting against its upper stop.\(^7\) Subtract that zero position from all of the position values measured during an experimental run. The total travel time up and down the ramp is then the difference in times between the first two vertical zero-crossings in the data.

The ratio of the total travel times for the sliding cart and the rolling cart is measured for 20 different ramp angles $\theta$, as plotted by the red dots in Fig. 4. More data are collected near the critical angle $\theta_c$ for which the ratio is unity (through which value the horizontal axis passes). The random variation in the data points is reflected in the error bars in the graph and represents practical limits in the reproducibility of the runs. At angles less than $\theta_c$, the rolling cart wins the race to return to its equilibrium position because friction significantly delays the sliding cart near its peak height. On the other hand, at track angles larger than $\theta_c$, the rolling cart loses the race because it wastes time climbing to a larger maximum height than does the sliding cart.

The two data curves that correspond to the point circled in green in Fig. 4 are graphed in Fig. 5. The curve for the sliding cart traveling up and down the track inclined at that $16^\circ$ angle has the same characteristic shape as those computed by numerical integration of Eqs. (2) and (5) by Heck and van Buuren.\(^8\)

The blue curve in Fig. 4 is the best fit to Eq. (8), yielding $\mu_{\text{fit}} = 0.20$ with a variation of $\pm 0.02$ to still pass through a reasonable number of experimental error bars.\(^9\) The coefficient of kinetic friction can be directly measured as the tangent of the angle—when Eq. (6) diverges—at which a cart with locked wheels just barely slides down the track at constant speed. It is found to be $\mu_{\text{exp}} = 0.165 \pm 0.005$ where the error bar reflects the uncertainty in trying to determine by how much the ramp angle needs to be decreased so that the cart slides down it at constant speed after being nudged into motion.

The fit and experimental values of $\mu$ do not quite agree to within their error bars. However, there are a number of systematic error sources that are not included in these random uncertainties. An important one is that the initial speed of the sliding cart is reduced by kinetic friction while the spring is decompressing. In addition, some translational mechanical energy is lost\(^{10}\) to speed-dependent air drag, to bowing and shaking of the track and stops, and to rolling friction on and rotational kinetic energy of the wheels of the rolling cart. Finally, the wheels are being repeatedly taped and untaped, and it is difficult to tape the wheels in exactly the same way each time.
References

5. Use a T10 star screwdriver to disassemble the cart. (The PASCO manual incorrectly says to use a number 1 Phillips screwdriver.) Remove the plunger assembly, the two magnets, and the four large nuts. See https://www.pasco.com/prodCatalog/ME/ME-6843_spring-cart-launcher/.
6. Th e zero position is measured immediately after each experimental run, to account for changes in the spring compression when the track angle is varied, as well as to account for any shifts in position of the motion detector, track, or spring stops during the firing and subsequent motion of the cart.
8. Equation (8) has a minimum at $\mu /\tan \theta = 0.3966$ corresponding to $\theta = 26.76^\circ$ for the blue curve in Fig. 4.
9. Since the two carts move at different average speeds, the energy loss will be slightly different for the two carts.
10. Three videos showing races between the rolling and sliding carts are available online at TPT Online under the Supplemental tab, http://dx.doi.org/10.1119/10.0004147.

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Conclusions

Two objects, differing only in whether or not they experience kinetic friction with the surface along which they move, are simultaneously launched up an inclined plane with the same initial speeds. Constant-acceleration kinematics and laboratory measurements both show that either object can return to its starting level first. Theory predicts that the winner depends on the value of the coefficient of friction $\mu$ for a fixed angle of incline $\theta$. Experimentally, however, it is simpler to vary the ramp angle $\theta$ for a fixed frictional coefficient $\mu$ (determined by the nature of the two surfaces in contact, namely here for the locked hard-plastic wheels of a laboratory cart sliding across an aluminum track). An identical cart with unlocked wheels rolls along the same track without resistive forces (ignoring rolling friction and air drag).

The essential reason that either the sliding or the rolling cart can win this race is that there are two competing factors at play. One is that the rolling cart travels faster at any given height and direction of motion than does the sliding cart. But the other is that the rolling cart climbs to a higher turnaround point than does the sliding cart. At low incline angles, the first factor dominates, and the sliding cart can be seen by eye to almost get stuck near its peak position along the track and thereby lose the race. On the other hand, at large angles of inclination, the normal force and therefore also the frictional force on the sliding cart get reduced to the point that the second factor dominates and it now wins the race. Supplementary movies show the red rolling cart winning at $\theta = 13^\circ$, the blue sliding cart winning at $\theta = 22^\circ$, and a tie at $\theta_c = 16^\circ$. 

Fig. 4. Each red point represents the ratio of travel times $T$ for the same cart measured twice on the same track, once with its wheels taped and once with its wheels freely turning. The error bars are the standard errors computed by Excel. The blue curve is the theoretical prediction from Eq. (8) for $\mu = 0.2$ and diverges to infinity at an angle of $\tan^{-1} 0.2 = 11.3^\circ$. On the other hand, if the horizontal axis is extended rightward to 90°, the blue curve slowly approaches unity (i.e., the horizontal axis) because the normal force in Fig. 1 (and hence the frictional force) becomes zero in that limit. Experimentally, however, the track angle is limited to about 40° by the achievable spring impulse.

Fig. 5. The pair of experimental runs corresponding to the red data point circled in green in Fig. 4. These data were collected using the motion detector at a rate of 30 samples per second. The vertical and horizontal starting point for each curve (as the cart first crosses through its equilibrium position after the spring pin is released) has been shifted to the origin.