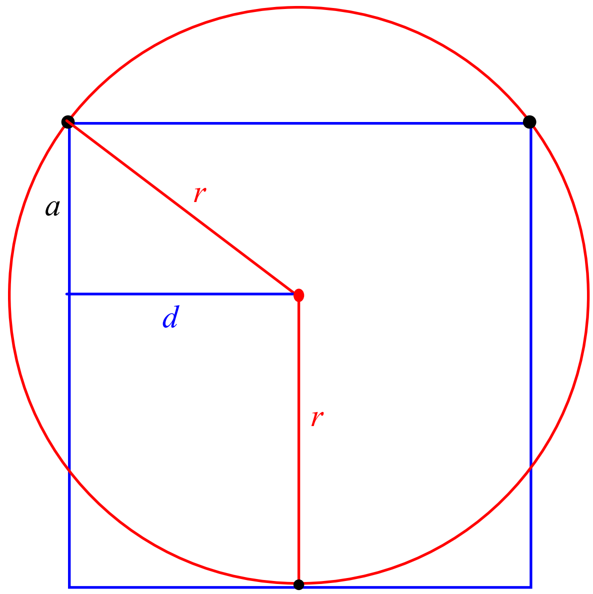
A 3-4-5 Triangle—C.E. Mungan, Spring 2020

Consider the following diagram showing a red circle and a blue square that intersect at the three black points.



The circle has radius r and the square has sides of half-length d. We see that the vertical height of the square can thus be written in two alternative ways as

$$a + r = 2d\,. \tag{1}$$

Now consider the triangle in this diagram. From Pythagoras theorem, we have

$$a^2 + \left(\frac{a+r}{2}\right)^2 = r^2 \quad \Rightarrow \quad 5a^2 + 2ra - 3r^2 = 0 \tag{2}$$

using Eq. (1), after multiplying the first equality through by 4. The positive root of this quadratic equation in a is $a/r = 0.6$ so that $d/r = 0.8$ from Eq. (1). Thus the triangle has sides in the ratio of 3-4-5, which is nifty.

We can now make some comparisons of the properties of the red circle and the blue square. For example, the square has slightly larger perimeter because

$$8d = 6.4r > 2\pi r \approx 6.28r\,. \tag{3}$$

On the other hand, the square has smaller area because

$$(2d)^2 = 2.56r^2 < \pi r^2 \approx 3.14r^2\,. \tag{4}$$