

Acceleration Components in 2D—C.E. Mungan, Fall 2000

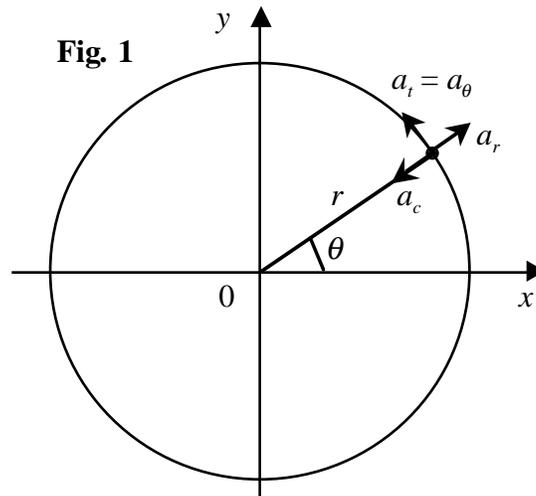
At any given instant in time, the trajectory of any object can be considered to lie on a portion of some circle. IF we choose the origin of our coordinate system to lie at the center of this circle, THEN the centripetal and radial directions are parallel,

$$a_r = -a_c \tag{1}$$

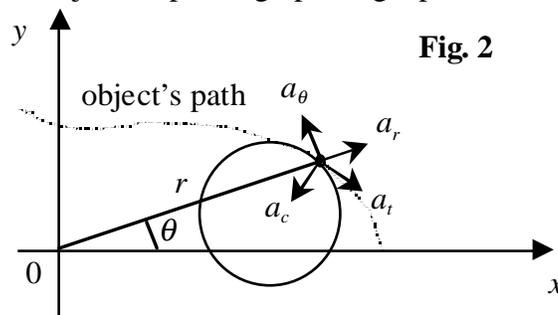
where the minus sign indicates the acceleration is INWARD toward the center of the circle, AND likewise the tangential and azimuthal directions are parallel,

$$a_\theta = \pm a_t \tag{2}$$

where the plus sign is used if the object is traveling in the conventional counter-clockwise sense and the minus sign for the opposite case. These two equations are easy to understand with the help of the following diagram (drawn assuming both ccw velocity and acceleration). See the Appendix for a further discussion of the sign conventions used here.



In contrast, IF the origin of the coordinate system does not coincide with the instantaneous center of curvature, THEN Eqs. (1) and (2) do NOT hold true. Here is an example where I have arbitrarily supposed that the object is speeding up along a path in the xy -plane toward the right.



In order to clarify the meaning of the four acceleration components in Fig. 2, consider a new origin $0'$ at the center of the indicated circle and denote the position of the particle relative to this origin by the plane polar coordinates r' and θ' . The primed azimuthal unit vector has

rectangular components,

$$\hat{\boldsymbol{\theta}}' = -\sin \theta' \hat{\mathbf{i}} + \cos \theta' \hat{\mathbf{j}} \quad (3)$$

so that its derivative is proportional to the primed radial unit vector,

$$\frac{d\hat{\boldsymbol{\theta}}'}{dt} = -\omega' (\cos \theta' \hat{\mathbf{i}} + \sin \theta' \hat{\mathbf{j}}) \equiv -\omega' \hat{\mathbf{r}}' \quad (4)$$

where the z -component of the object's primed angular velocity is

$$\omega' \equiv \frac{d\theta'}{dt} = \frac{v'_\theta}{r'} \quad (5)$$

By definition the object's velocity is tangential,

$$\mathbf{v} = v'_\theta \hat{\boldsymbol{\theta}}' \quad (6)$$

(Here v'_θ and ω' are both signed, being negative for cw rotations such as in Fig. 2, whereas the object's speed, $v = v_r$, is not.) The time derivative of Eq. (6) defines the object's acceleration,

$$\mathbf{a} = \frac{dv'_\theta}{dt} \hat{\boldsymbol{\theta}}' - \frac{v'^2_\theta}{r'} \hat{\mathbf{r}}' \quad (7)$$

where Eqs. (4) and (5) was used to rewrite the last term. Note that the first term on the right is tangential, while the second is centripetal. We have thus derived the well-known results,

$$a_t = \frac{dv}{dt} \quad \text{and} \quad a_c = \frac{v^2}{r'} \quad (8)$$

The tangential acceleration is determined by how the *magnitude* of the object's velocity changes, while the centripetal acceleration is determined by how the *direction* of the velocity varies.

In contrast, the derivative of the unprimed position vector, $\mathbf{r} = r\hat{\mathbf{r}}$, is

$$\mathbf{v} = \frac{dr}{dt} \hat{\mathbf{r}} + r\omega \hat{\boldsymbol{\theta}} \equiv v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} \quad (9)$$

Here the azimuthal component follows from the analogs of Eqs. (3)–(5),

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt} \hat{\boldsymbol{\theta}} \equiv \omega \hat{\boldsymbol{\theta}} \quad (10)$$

Differentiating Eq. (9) now results in

$$\mathbf{a} = (\dot{v}_r \hat{\mathbf{r}} + v_r \omega \hat{\boldsymbol{\theta}}) + (v_r \omega \hat{\boldsymbol{\theta}} + r\alpha \hat{\boldsymbol{\theta}} - r\omega^2 \hat{\mathbf{r}}) \quad (11)$$

where the first parenthesis contains the time derivative of $v_r \hat{\mathbf{r}}$, and the second parenthesis the derivative of $r\omega \hat{\boldsymbol{\theta}}$ with $\alpha \equiv d\omega / dt$. Rearranging we see that

$$a_r = \dot{v}_r - r\omega^2 \quad \text{and} \quad a_\theta = r\alpha + 2v_r \omega \quad (12)$$

The radial acceleration is determined by the time derivative of the radial velocity plus a centrifugal term, whereas the azimuthal acceleration is determined by the angular acceleration

plus a Coriolis term.

To summarize, one can choose either to work with the (centripetal, tangential) pair of orthogonal components defined by Eq. (8) or the (radial, azimuthal) pair of Eq. (12). Textbooks such as Serway, however, invite great confusion by mixing the nomenclature and referring to Eq. (8) as a (radial, tangential) pair. ONLY for the special case of circular motion does it make sense to use the corresponding terms interchangeably, and even then the danger of sign confusion for students is real.

Appendix—Meaning of the adjective “tangential”

There are two possible ways to define the adjective “tangential.” One way is to consider it to be a synonym for “azimuthal” relative to the instantaneous circle of curvature (i.e., the tangential direction is defined to be $\hat{\theta}'$). In that case, the sign in Eq. (2) is ALWAYS positive. I have chosen the other convention, which is more popular with introductory textbooks, namely that the tangential direction is in the direction of motion of the object (regardless of whether it moves cw or ccw). That is, v_t is always positive, while a_t is positive if the object is speeding up and negative if slowing down. It is left as an exercise to the reader to verify that while the sign in Eq. (2) depends (in the manner already discussed) on which of these two definitions is adopted, $a_t = \dot{v}_t$ for EITHER convention.