Relation between the Ampère-Maxwell and Biot-Savart Laws—C.E. Mungan, Spring 2005

This document is a discussion of Michael Langham's poster on "Maxwell's Law" presented as paper AB02 on 1/10/05 at the AAPT meeting in Albuquerque. I extend his results to a charge distribution rather than a single point charge, allow for Ampère's source term in the Maxwell law, and clarify what exactly Michael's result proves.

Suppose there is a charge distribution in space with (volume) charge density ρ. Next suppose there is a small platform located at position $\mathbf{r}$, at which we are interested in measuring the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, respectively. Consider three different possibilities for the motion of the charges and platform:

A—The charges and platform are at rest;

B—The platform is at rest, but the charges are all moving with the same velocity $\boldsymbol{\upsilon}$;

C—Both the platform and charges are moving with velocity $\boldsymbol{\upsilon}$.

Clearly configurations A and C must be relativistically identical, so that $d\mathbf{E}/dt = 0$ is measured at the platform in both cases. But for configuration C we can partition this time dependence of $\mathbf{E}(\mathbf{r},t)$ into two parts,

$$\frac{d\mathbf{E}}{dt} = \frac{\partial \mathbf{E}}{\partial t} + \left(\frac{\partial \mathbf{E}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{E}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{E}}{\partial z}\frac{dz}{dt} \right) \tag{1}$$

which becomes

$$0 = \frac{\partial \mathbf{E}}{\partial t} + (\boldsymbol{\upsilon}\bullet\nabla)\mathbf{E}, \tag{2}$$

often called the "material derivative."

But now consider the vector identity (obtained from the BAC-CAB rule for constant vector field $\boldsymbol{\upsilon}$)

$$\nabla \times (\boldsymbol{\upsilon} \times \mathbf{E}) = \boldsymbol{\upsilon}(\nabla\bullet\mathbf{E}) - (\boldsymbol{\upsilon}\bullet\nabla)\mathbf{E} = \boldsymbol{\upsilon}\frac{\rho}{\epsilon_0} - (\boldsymbol{\upsilon}\bullet\nabla)\mathbf{E}, \tag{3}$$

using Gauss' law in the second step. Solve Eq. (2) for the last term and substitute that in for the last term in Eq. (3) to obtain

$$\nabla \times (\boldsymbol{\upsilon} \times \mathbf{E}) = \boldsymbol{\upsilon}\frac{\rho}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}. \tag{4}$$

Finally divide this equation by the speed of light squared to get

$$\nabla \times \left(\frac{\boldsymbol{\upsilon} \times \mathbf{E}}{c^2} \right) = \mu_0(\rho\boldsymbol{\upsilon}) + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}. \tag{5}$$

We immediately recognize this as the Ampère-Maxwell law provided that the current density is identified as

$$\mathbf{J} = \rho \boldsymbol{\upsilon} \tag{6}$$

and the magnetic field produced at position **r** in configuration B is

$$\mathbf{B} = \frac{\boldsymbol{\upsilon} \times \mathbf{E}}{c^2}. \tag{7}$$

It is easy to verify that Eq. (6) correctly describes our charge distribution in uniform motion. Define the x-axis to be the direction of the velocity $\boldsymbol{\upsilon}$. Then,

$$\mathbf{J} = J\hat{\mathbf{i}} \equiv \frac{dQ}{(dydz)dt}\hat{\mathbf{i}} = \frac{dQdx}{(dxdydz)dt}\hat{\mathbf{i}} = \frac{dQ}{dV}\frac{dx}{dt}\hat{\mathbf{i}} = \rho\boldsymbol{\upsilon} \tag{8}$$

as desired. Furthermore we can readily show that Eq. (7) is simply the Biot-Savart law in disguise,

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{(\mathbf{J}dV)\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{\upsilon}\times\hat{\mathbf{r}}}{r^2}\rho dV = \frac{\boldsymbol{\upsilon}}{c^2}\times\left(\frac{dQ}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\right) = \frac{\boldsymbol{\upsilon}\times d\mathbf{E}}{c^2} \tag{9}$$

using Coulomb's law in the last step.

To conclude, the present derivation shows that in magnetostatics the Ampère-Maxwell and Biot-Savart laws are equivalent (i.e., if you assume either one, you can derive the other one), in the same way that in electrostatics Gauss' and Coulomb's law are equivalent.