Speeds at Perigee and Apogee given the Radial Distances—C.E. Mungan, Fall 2022

Suppose we know the radial distances r at apogee (denoted by subscript "a") and perigee (subscript "p") of a low-mass satellite in an elliptical orbit about the earth. Then we can compute the speeds υ at those two points in the orbit by simultaneously conserving mechanical energy and angular momentum.

Since the satellite has low mass, we can neglect any motion of the earth and approximate the center of mass of the system of earth and satellite to be located at earth's center, which we choose to be the origin of our coordinate system and is at a focus of the ellipse. In that case, the mechanical energy of the system is equal at apogee and perigee so that

$$\frac{1}{2}m\upsilon_\text{a}^2 - \frac{GMm}{r_\text{a}} = \frac{1}{2}m\upsilon_\text{p}^2 - \frac{GMm}{r_\text{p}} \quad \Rightarrow \quad \upsilon_\text{a}^2 = \upsilon_\text{p}^2 + 2GM\left(\frac{1}{r_\text{a}} - \frac{1}{r_\text{p}}\right) \tag{1}$$

where $G = 6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$ is the universal gravitational constant, $M = 5.97\times10^{24}\ \text{kg}$ is the mass of the earth, and $m \ll M$ is the mass of the satellite. At apogee and perigee the velocity of the satellite is perpendicular in direction to its radial position, so that conservation of angular momentum between these two points in the orbit becomes simply

$$m\upsilon_\text{a}r_\text{a} = m\upsilon_\text{p}r_\text{p} \quad \Rightarrow \quad \upsilon_\text{a}^2 = \frac{r_\text{p}^2}{r_\text{a}^2}\upsilon_\text{p}^2 . \tag{2}$$

Equate the right-hand sides of Eqs. (1) and (2) to get

$$\upsilon_\text{p}^2\left(\frac{r_\text{p}^2}{r_\text{a}^2} - 1\right) = 2GM\left(\frac{1}{r_\text{a}} - \frac{1}{r_\text{p}}\right) \quad \Rightarrow \quad \upsilon_\text{p}^2\,\frac{(r_\text{p} - r_\text{a})(r_\text{p} + r_\text{a})}{r_\text{a}^2} = 2GM\,\frac{r_\text{p} - r_\text{a}}{r_\text{a}r_\text{p}} \tag{3}$$

which gives the solutions

$$\upsilon_\text{p} = \sqrt{\frac{2GMr_\text{a}}{(r_\text{p} + r_\text{a})r_\text{p}}} \tag{4}$$

and, as expected by interchanging the two subscripts,

$$\upsilon_\text{a} = \sqrt{\frac{2GMr_\text{p}}{(r_\text{p} + r_\text{a})r_\text{a}}} . \tag{5}$$

For example, if $r_\text{p} = 6870\ \text{km}$ and $r_\text{a} = 45\,400\ \text{km}$, then these solutions imply $\upsilon_\text{p} = 10.0\ \text{km/s}$ and $\upsilon_\text{a} = 1.52\ \text{km/s}$. Multiplying together Eqs. (4) and (5) gives the symmetric result

$$\upsilon_\text{a}\upsilon_\text{p} = \frac{GM}{a} \tag{6}$$

where the semi-major axis is $a = (r_\text{p} + r_\text{a})/2$. Equation (6) says the *geometric* average of the apsidal speeds is given by the familiar formula for speed in a circular orbit whose radius equals the *arithmetic* average of the apsidal distances,

$$\upsilon_{\text{avg}} = \sqrt{\frac{GM}{r_{\text{avg}}}} \,. \tag{7}$$

If we know this equation (which admittedly requires remembering which average is geometric and which is arithmetic) then we can combine it with Eq. (2) to quickly reproduce Eqs. (4) and (5).