

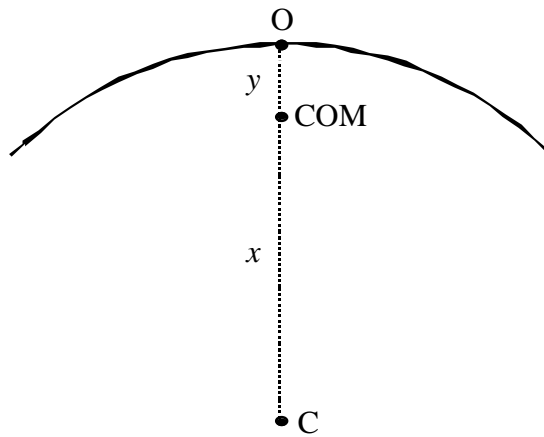
Small-Angle Oscillations of an Arc about its Midpoint—C.E. Mungan, Summer 2000

Prove that for small angular displacements, a uniform circular arc of radius R balanced on a knife edge executes simple harmonic motion of period

$$T = 2\pi \sqrt{\frac{2R}{g}} \quad (1)$$

regardless of the mass of the arc and of what fraction of a complete circle that it is.

Denote the center of the circle of which the arc is a portion by C , the suspension point by O , and the center of mass of the arc by COM , as sketched below. Also let x be the distance from C to the COM and let y be the distance from the COM to O , so that $x + y = R$.



The moment of inertia about C is trivial to find since all points on the arc are equidistant from the circular center,

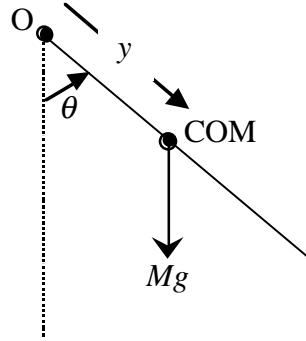
$$I_C = MR^2 = I_{\text{COM}} + Mx^2 \quad (2)$$

where M is the mass of the arc and I used the parallel-axis theorem in the second step. Using the parallel-axis theorem again to find the moment of inertia about the point of oscillations gives

$$I_O = I_{\text{COM}} + My^2 = M(R^2 - x^2 + y^2) = 2MRy \quad (3)$$

where I replaced I_{COM} using Eq. (2) in the second step and then used the fact that $x = R - y$ in the last step.

Let's quickly remind ourselves of how to find the angular frequency ω for simple harmonic motion of a physical pendulum by inspection of the following diagram.



The torque about the point of oscillations is related to the rotational analog of Newton's second law,

$$\begin{aligned}\tau_O &= -Mgy \sin \theta \cong -Mgy\theta \\ &= I_O \ddot{\theta} = -I_O \omega^2 \theta\end{aligned}\tag{4}$$

so that

$$\omega \equiv \frac{2\pi}{T} = \sqrt{\frac{Mgy}{I_O}}.\tag{5}$$

Substituting Eq. (3) and rearranging leads immediately to Eq. (1). The length of an equivalent simple pendulum is thus the diameter of the circle even if the arc is so short (albeit still long compared to its radial thickness) that it hardly looks circular, a not particularly intuitive result!