Model for the Atmosphere—C.E. Mungan, Summer 2002

The following atmospheric model taken from TPT **36**:288 (1998) is in reasonable agreement with actual average values up through the troposphere. It uses more realistic assumptions than the typical isothermal model which predicts an exponential dependence of pressure on altitude *z*, and yet it remains soluble using basic calculus.

It is based on three assumptions. First, we assume that any parcel of moving air flows adiabatically from one location to another. Hence, according to the first law of thermodynamics,

$$dQ = C_P dT - V dP = 0 \quad \Rightarrow \quad \frac{C_P}{V} = \frac{dP}{dz}\frac{dz}{dT} \tag{1}$$

where C_P is the isobaric heat capacity. Secondly, we approximate the atmosphere as an ideal diatomic gas, composed of 79% nitrogen and 21% oxygen. From the ideal gas law, we have

$$P = \frac{\rho RT}{M} \tag{2}$$

where the molar mass is $M = 0.79 \times 28\ \text{g}/\text{mol} + 0.21 \times 32\ \text{g}/\text{mol}$, and the heat capacity (at terrestial temperatures) is

$$C_P = \tfrac{7}{2} nR. \tag{3a}$$

Finally, we assume that each slab of the atmosphere is always approximately in mechanical equilibrium, so that the buoyant force balances the weight,

$$dP = -\rho g dz. \tag{4}$$

The minus sign anticipates the fact that the pressure decreases with height.

We solve these equations simultaneously to deduce how the temperature, density, and pressure vary with height. The first is the easiest to solve for. Substitute Eq. (4) into (1),

$$\frac{m}{m}\frac{C_P}{V} \equiv \rho c_P = -\rho g \frac{dz}{dT} \quad \Rightarrow \quad \frac{dT}{dz} = -\frac{g}{c_P}, \tag{5}$$

where the specific heat is

$$c_P \equiv \frac{C_p}{m} = \frac{7R}{2M}, \tag{3b}$$

while the molar heat capacity is

$$c'_P \equiv \frac{C_p}{n} = 3.5R. \tag{3c}$$

Note that Eq. (5) predicts a linear temperature profile of –9.7 ˚C/km, i.e.,

$$\int_{T_0}^{T} dT = -\frac{g}{c_P}\int_0^z dz \quad \Rightarrow \quad \boxed{\frac{T}{T_0} = 1 - \frac{z}{z_0}}. \tag{6}$$

The sea level values are taken to be $T_0 = 293\ \text{K}$, $P_0 = 101.3\ \text{kPa}$, and $z_0 \equiv c_P T_0 / g = 30.1\ \text{km}$.

To get the density, substitute Eq. (6) into (2),

$$P = \frac{\rho R T_0}{M}\left(1 - \frac{z}{z_0}\right).$$

Take the differential and equate it to Eq. (4) to get

$$-\rho g dz = \frac{RT_0}{M}\left(1-\frac{z}{z_0}\right)d\rho - \frac{\rho R T_0}{M z_0}dz \quad \Rightarrow \quad \left(1-\frac{c_P'}{R}\right)\int_0^z \frac{dz}{z_0 - z} = \int_{\rho_0}^{\rho}\frac{d\rho}{\rho}$$

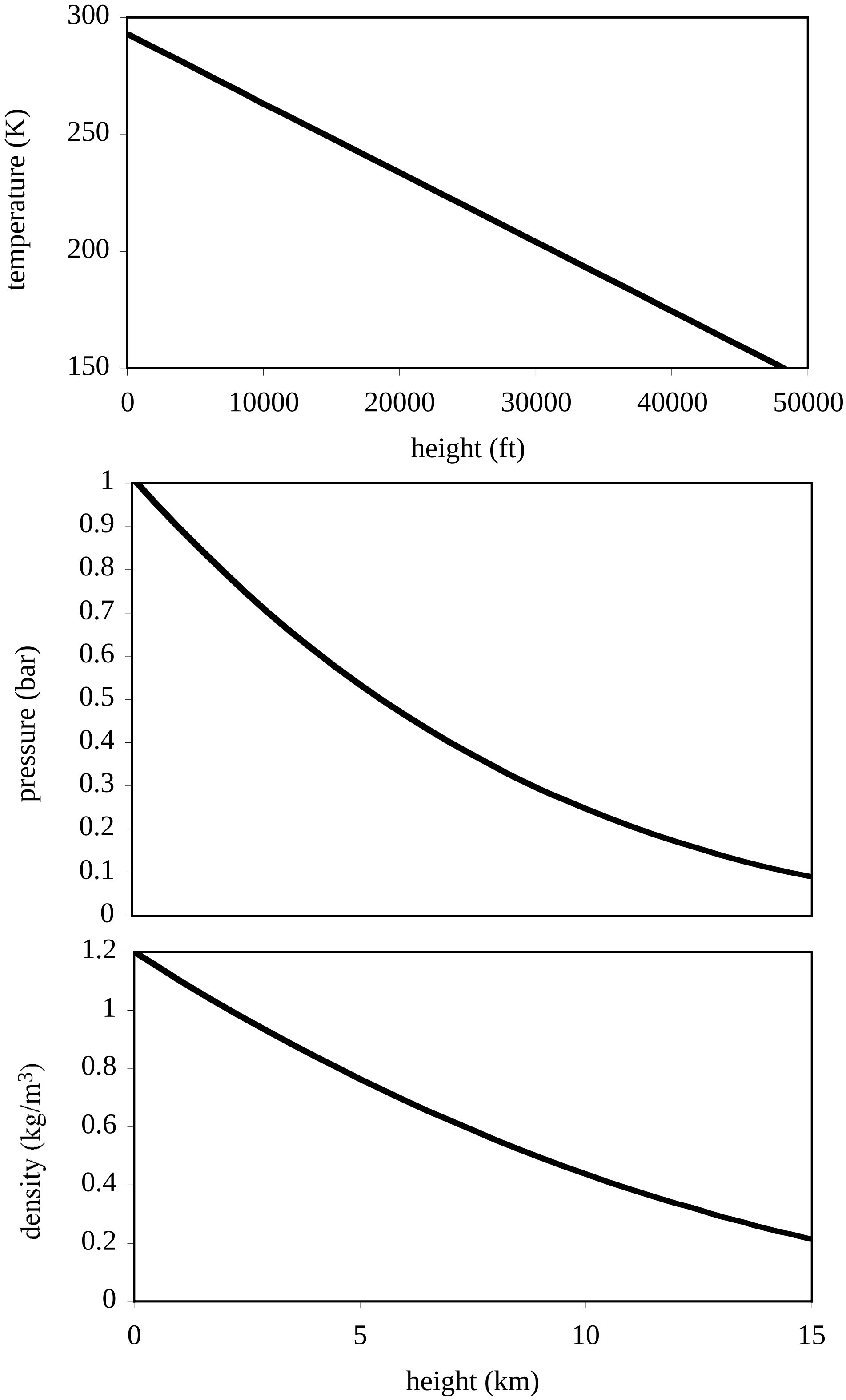
$$\Rightarrow \quad \left(\frac{c_P'}{R}-1\right)\ln\left(1-\frac{z}{z_0}\right) = \ln\frac{\rho}{\rho_0} \quad \Rightarrow \quad \boxed{\frac{\rho}{\rho_0} = \left(1-\frac{z}{z_0}\right)^{c_P'/R-1}} \tag{7}$$

where $\rho_0 \equiv MP_0 / RT_0 = 1.20\ \text{kg/m}^3$, in good agreement with the standard value of $(1.29\ \text{kg/m}^3)\times(273\ \text{K})/(293\ \text{K})$.

Finally, substitute Eqs. (6) and (7) into (2) to obtain

$$\boxed{\frac{P}{P_0} = \frac{\rho}{\rho_0}\frac{T}{T_0} = \left(1-\frac{z}{z_0}\right)^{c_P'/R}}. \tag{8}$$

Equations (6)–(8) are plotted on the same horizontal axis on the following page. Note that atmospheric pressure has dropped by about 25% at 7600 ft, the elevation of Los Alamos, NM. From personal experience, I can testify that any sealed bottles brought there from sea level need to be unscrewed gingerly!


temperature (K)
300
250
200
150
0
10000
20000
30000
40000
50000
height (ft)
pressure (bar)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
density (kg/m3)
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
height (km)