Bead Sliding on a Rotating Rod with Linear Drag—C.E. Mungan, Summer 2021

A frictionless rod is forced to rotate at constant angular speed ω about the point $r = 0$. A small bead is initially located at that point at rest (in unstable equilibrium). It is given an impulsive radial speed of υ_0 at $t = 0$. It slides outward along the rod while subject to air drag that is linear in the speed of the bead. Find the subsequent position $r(t)$ of the bead.

Apply Newton's second law in the rotating reference frame of the rod. The net azimuthal force on the bead must be zero since its angular speed ω is constant; thus the normal force of the rod on the bead is equal and opposite to the sum of the Coriolis force and azimuthal component of the drag force. In the radial direction, the acceleration results from the sum of the centrifugal force and radial component of the drag force so that

$$m\ddot{r} = m\omega^2 r - 2\gamma m\dot{r} \tag{1}$$

where m is the mass of the bead and γ is the drag coefficient (in units of s^{-1}). Thus

$$\ddot{r} + 2\gamma\dot{r} - \omega^2 r = 0 \tag{2}$$

which is identical to the equation of a 1D damped undriven oscillator <u>except</u> for the minus sign in front of the third term (because we no longer have a restoring force). We guess that this opposite sign changes the trigonometric into hyperbolic functions so that

$$r = e^{-\gamma t}(A\cosh\Omega t + B\sinh\Omega t) \tag{3}$$

whose time derivatives are

$$\dot{r} = -\gamma e^{-\gamma t}(A\cosh\Omega t + B\sinh\Omega t) + \Omega e^{-\gamma t}(A\sinh\Omega t + B\cosh\Omega t) \tag{4}$$

and

$$\ddot{r} = -2\Omega\gamma e^{-\gamma t}(A\sinh\Omega t + B\cosh\Omega t) + (\Omega^2 + \gamma^2)e^{-\gamma t}(A\cosh\Omega t + B\sinh\Omega t)\,. \tag{5}$$

Substitute Eqs. (3) through (5) into (2) and separately equate coefficients of $e^{-\gamma t}\cosh\Omega t$ and $e^{-\gamma t}\sinh\Omega t$ to find

$$\Omega = \sqrt{\omega^2 + \gamma^2} \tag{6}$$

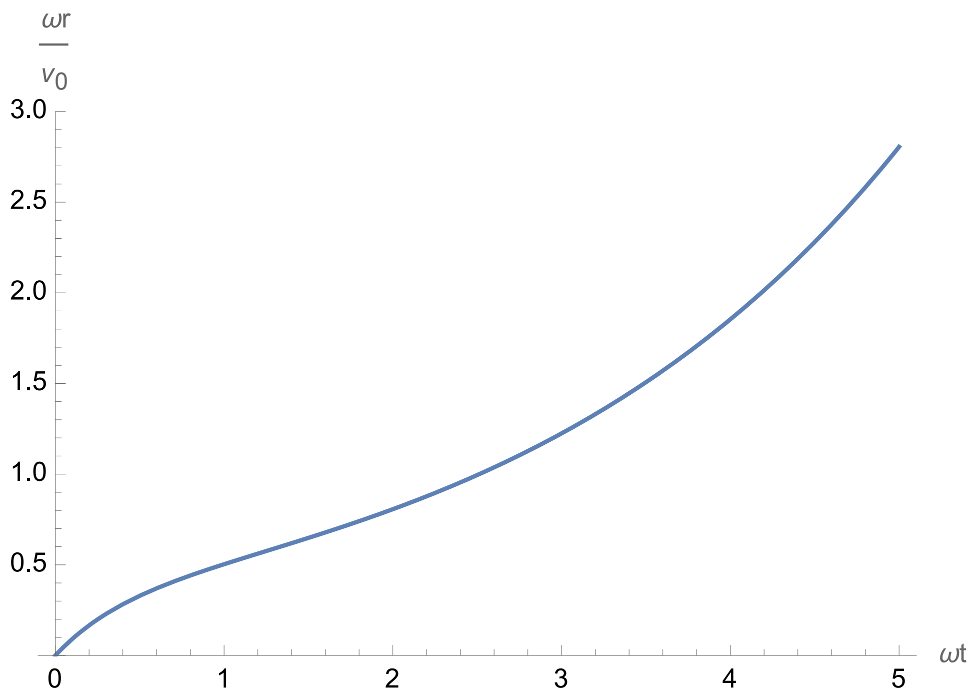
which is larger than ω, in contrast to the *reduced* angular frequency $(\omega^2 - \gamma^2)^{1/2}$ of an underdamped oscillator. Finally, fit Eqs. (3) and (4) to the initial conditions to get

$$r = \frac{\upsilon_0}{\Omega}e^{-\gamma t}\sinh\Omega t\,. \tag{7}$$

This expression can be written in the dimensionless form $R \equiv \omega r/\upsilon_0$ as a function of $T \equiv \omega t$ in terms of the single parameter $\Gamma \equiv \gamma/\omega$,

$$R = \frac{e^{-\Gamma T}}{\sqrt{1+\Gamma^2}}\sinh\sqrt{1+\Gamma^2}\,T\,. \tag{8}$$

For example, the graph on the top of the next page plots the radial displacement as a function of time for the case of $\Gamma = 1 \Rightarrow \gamma = \omega$.



For small t (compared to the smaller of ω^{-1} and γ^{-1}), Eq. (7) becomes

$$\frac{r}{v_0} \approx t - \gamma t^2 \tag{9}$$

which is why the above graph is initially parabolic downward. However, for large t, Eq. (8) becomes

$$2R\sqrt{1+\Gamma^2} \approx e^{\Gamma T[(1+\Gamma^{-2})^{1/2}-1]} \tag{10}$$

which is the exponential of a positive constant multiplied by time, and thus the graph must turn around and diverge to infinity as observed above.